Trends and Compensation

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Mathematics Colloquium
March 20, 2003

We live in an uncertain world

- Earthquakes endanger millions of lives in California
- Presumed benefits of early cancer detection are called into question
- More than 950 bankruptcies in the US over the past three decades cost society trillions of Dollars

Accurate forecasts of event risk can

- Tell us when to evacuate
- Distinguish between useful and useless medical screening
- Equitably distribute risk of default (instead of concentrating it on Enron employees)

Forecasting event risk

- All the events mentioned above happen for a reason
  - Cause and effect models
- All the events mentioned above can be analyzed statistically
  - Intensity based models
- These approaches carry complementary information so it is desirable to combine them
  - But it is not necessarily easy

Classical theory of stochastic processes has a lot to offer

uncertainty	filtration
evolution of information	probability space
event arrival	submartingale
conditional event probability	compensator
event insurance premium	stopping time

Cause and effect model of default

- Given is a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$
- A stopping time $T : \Omega \to [0, \infty]$ representing default is created from firm-specific information
  - **Example:** a firm defaults if the market value of its assets V falls below its liabilities L at their maturity τ (Black-Scholes (1973), Merton (1974)):

$$T = \tau \quad \text{if} \quad V_\tau \leq L \quad \text{and} \quad T = \infty \quad \text{otherwise}$$

  - **Example:** a firm defaults if V falls below L for the first time (Black-Cox (1976)):

$$T = \inf\{t > 0 : V_t \leq L\}$$

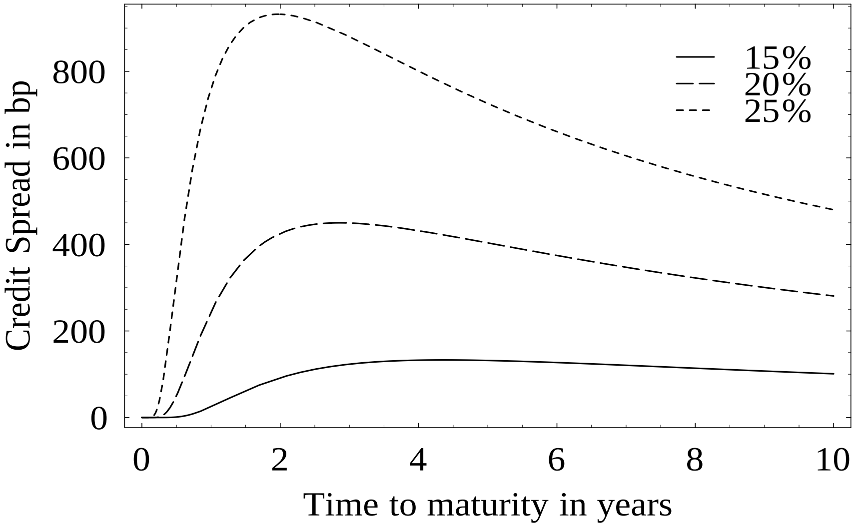
Can we predict the future?

- A stopping time T on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is called *predictable* if there exist $(T_n)_{n\geq 1}$ such that $(T_n) \uparrow T$ almost surely
  - **Example:** asset process V is a Brownian motion; then an announcing sequence is $T_n = \inf\{t > 0 : V_t \leq L + 1/n\}$
  - The associated *indicator process* $Y_t = 1_{\{T\leq t\}}$ is predictable as well: $Y_t \in \mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$
- Consider the *credit spread* $S(t) = -\frac{1}{t}\log P[T > t]$ for $t > 0$

  If T is predictable, we have

$$\lim_{t\downarrow 0} S(t) = 0$$

Credit spreads in the cause and effect model



- Cause and effect default models based on predictable stopping times are not plausible
- What can we do?
  - Introduce more complicated asset process
  - Try something else entirely

Reduced form model of default

- Given is a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$
- A stopping time $T : \Omega \to [0, \infty]$ representing default is constructed from a conditional default rate λ, which satisfies

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} P[T \leq t + h \,|\, \mathcal{F}_t], \quad t < T$$

  almost surely (Jarrow-Turnbull (1995), Duffie-Singleton (1999))

  - **Example:** Poisson process with (constant) intensity λ

- The existence of an intensity implies that the stopping time T is *unpredictable*: $P[T = S < \infty] = 0$ for all predictable times S

Credit spreads in the reduced form approach

- Credit spreads are bounded away from zero:

$$\lim_{t \downarrow 0} S(t) = \lambda_0$$

  i.e. short spreads are given by the intensity

- Caveats:
  - Economic intuition?
  - Existence of intensity λ_t?

Processes that, on average, never decrease

- A $\boxed{\text{submartingale}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is an adapted process Y such that each Y_t is integrable and

$$Y_s \leq E[Y_t | \mathcal{F}_s], \quad s \leq t,$$

  meaning that Y has a non-downward trend

  - **Example:** $Y_t = 1_{\{T \leq t\}}$ with natural filtration $(\mathcal{F}_t)$
  - **Example:** $Y_t = E[1_{\{T \leq t\}} \,|\, \mathcal{F}_t]$ for some filtration $(\mathcal{F}_t)$
  - **Example:** Y increasing, adapted to some filtration $(\mathcal{F}_t)$

- A submartingale is a generalization of a *martingale* M

$$M_s = E[M_t | \mathcal{F}_s], \quad s \leq t$$

Isolating the trend

Theorem 1 (Doob (1953), Meyer (1962, 1963)) *Let Y be a bounded positive submartingale.*

Then there exists a predictable, integrable, non-decreasing process A with $A_0 = 0$ such that

$$Y = M + A$$

where M is a martingale.

The process A is unique and called the $\boxed{\text{compensator}}$ *of Y.*

Understanding the compensator

- A satisfies

$$A_t = \lim_{h \downarrow 0} \frac{1}{h} \int_0^t \Big(E[Y_{s+h} \,|\, \mathcal{F}_s] - Y_s \Big) ds,$$

  or, in a heuristic sense, $dA_t = E[dY_t \,|\, \mathcal{F}_t]$

- **Example:** The compensator of a Poisson process with intensity λ with respect to its natural filtration is given by $A_t = \lambda \cdot t$
  - Here A is even absolutely continuous with respect to Lebesgue measure

When is the compensator continuous?

- Consider the indicator submartingale $Y_t = 1_{\{T \le t\}}$
- What happens to A at the jump time T?
  - T predictable: Y is predictable, increasing, so that $A = Y$
  - T unpredictable: $P[T = S < \infty] = 0$ for all predictable times S
    - $\Delta Y_S = Y_S - Y_{S-} = 0$ almost surely
    - $\Delta A_S = E[\Delta Y_S \,|\, \mathcal{F}_{S-}] = 0$ almost surely
    - Since A is predictable, it is continuous almost surely

Regular submartingales

A submartingale Y is *regular* if for all $(S_n) \uparrow S$

$$\lim_{n \to \infty} E[Y_{S_n}] = E[Y_S]$$

- **Example:** If Y is continuous, then it is regular
- **Example:** If T is unpredictable, then $Y_t = 1_{\{T \leq t\}}$ is regular

Theorem 2 *The compensator of a submartingale Y is continuous if and only if Y is regular.*

- Even if Y is regular, its compensator A need not be absolutely continuous with respect to Lebesgue measure: there may not exist an intensity λ such that

$$A_t = \int_0^t \lambda_s ds$$

Compensating for default

- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ for $\boxed{\text{uncertainty}}$
- Cause and effect model for $\boxed{\text{default time}}$

$$T = \inf\{t > 0 : V_t \leq L\}$$

for given

- $V =$ the market value of the firm, and
- $L =$ the book value of the firm's liabilities

What do investors really know?

- They may have access to stock prices, balance sheets, credit ratings, etc. The corresponding $\boxed{\text{information}}$ filtration $(\mathcal{F}_t)$ can look quite differently:
  - **Example:** $\mathcal{F}_t = \{\emptyset, \Omega\}$
  - **Example:** $\mathcal{F}_t = \sigma(V_s + U_s : s \leq t, s \in \mathcal{S})$ with U being "noise"
  - **Example:** $\mathcal{F}_t = \sigma(V_s : s \leq t)$
- They do not precisely know the firm's liabilities L
- They observe the firm's default state

Insuring events

- *Event-contingent claim* (T, Z, m) for the event time T, where
  - $Z \in \mathcal{G}_m$ is a promised random payoff, and
  - m is a maturity date at which the contract pays $Z \cdot 1_{\{T>m\}}$
- We are interested in the fair value of the event-contingent claim

The fair price of event insurance

Assume interest rates are zero.

Theorem 3 *Let Y be the conditional default probability $Y_t = E[1_{\{T \le t\}} \,|\, \mathcal{F}_t]$ and A its compensator. Define the* pricing trend

$$K_t = \int_0^t \frac{dA_s}{1 - Y_{s-}}$$

If Y is regular, then the actuarially fair value of the event-contingent claim (T, Z, m) at time zero is given by

$$E[Z \cdot e^{-K_m}]$$

Event survival probabilities are given by

$$P[T > t] = E[e^{-K_t}]$$

Case study

- Assume that investors observe assets V perfectly: $\mathcal{F}_t = \sigma(V_s : s \leq t)$
- Supposing L is independent of V and has distribution function G,

$$Y_t = E[1_{\{T \leq t\}} \,|\, \mathcal{F}_t] = P[M_t \leq L \,|\, \mathcal{F}_t] = 1 - G(M_t),$$

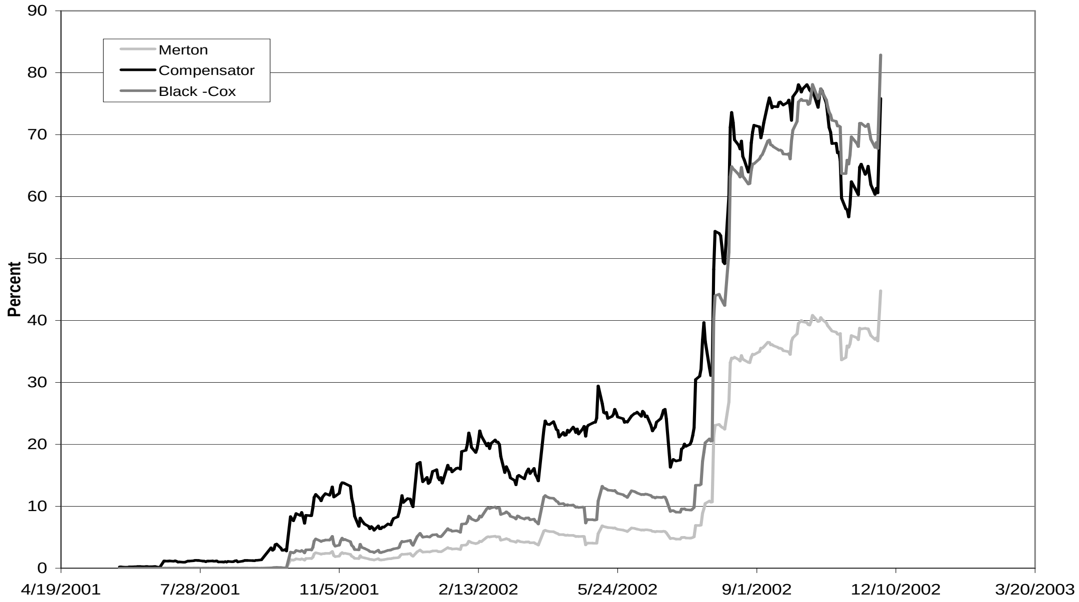
  where $M_t = \min_{s \leq t} V_s$ is the running minimum of V

- If V is continuous, then $K = Y$ and the pricing trend is

$$A_t = \int_0^t \frac{dY_s}{1 - Y_s} = -\log(1 - Y_t) = -\log G(M_t)$$

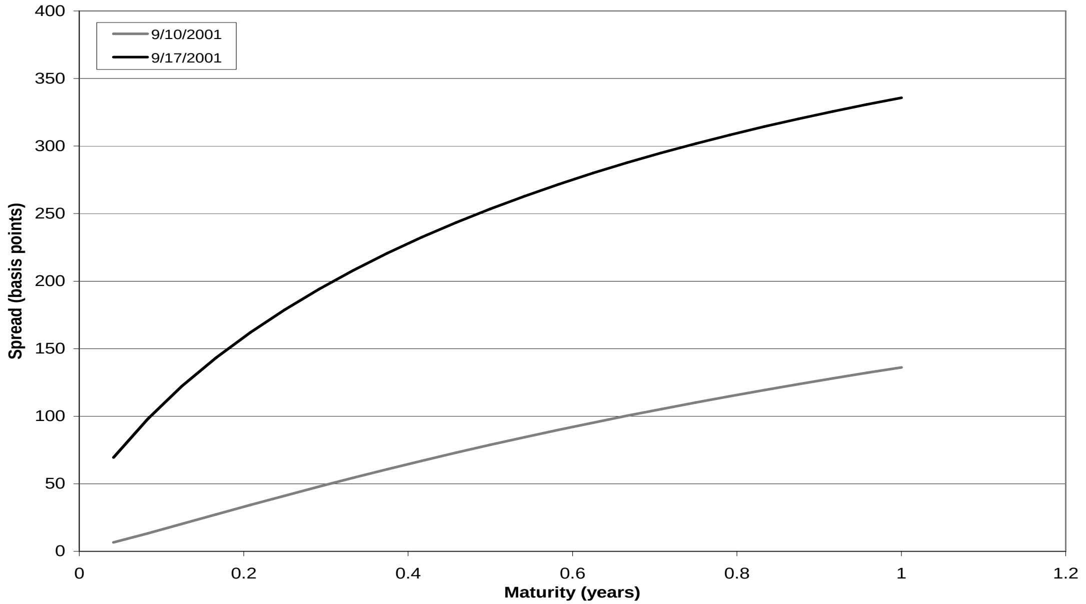
- Since $\mu\{t > 0 : V_t = M_t\} = 0$, K is *singular* with respect to Lebesgue measure μ, so an intensity does not exist

United Airlines default probability forecasts



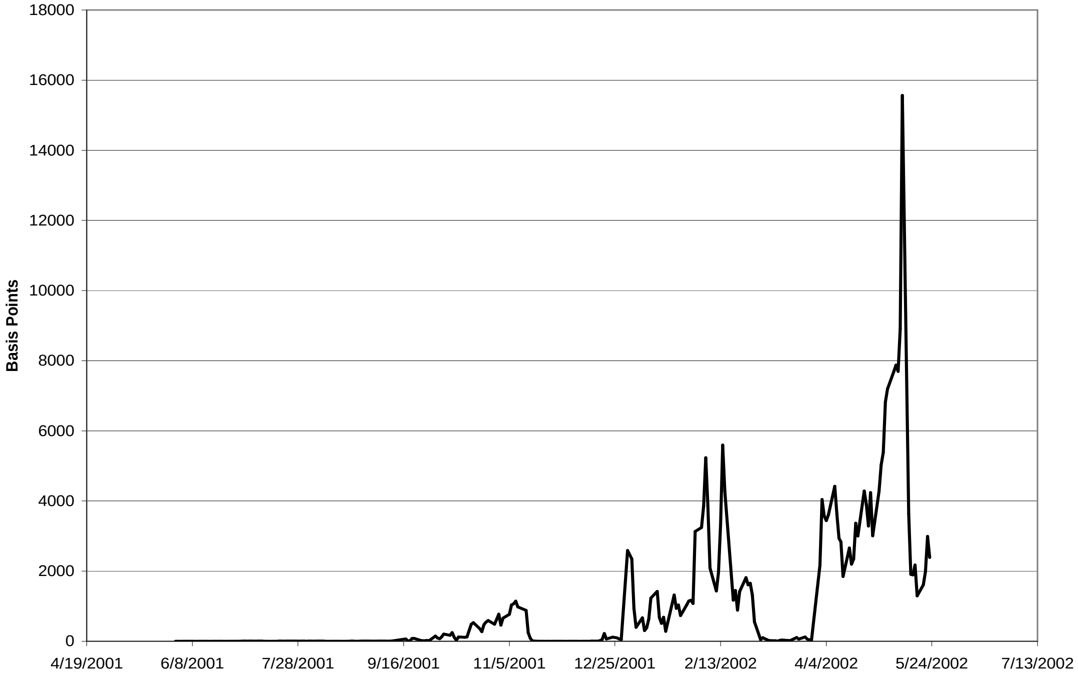
- Thanks to the uncertainty regarding the firm's liabilities, the compensator model produces realistically sized spreads
- The compensator model reacts more quickly since it takes account of the entire history of public information

United Airlines credit spreads



- On September 17, 2001, the first trading day after the WTC was attacked, UAL stock fell more than 40% to a new low
- The figure shows the term structure of credit spreads before and after the crisis

United Airlines short credit spreads



- Due to the unpredictability of the default stopping time the compensator model predicts positive spreads for firms in distress
- With a predictable default time the short spread is always zero

Conclusions

- The coherent integration of cause and effect event models and statistical reduced-form models is facilitated with compensators, which are well-known in the classical theory of stochastic processes
- There is a relationship between analytical properties of compensators and probabilistic properties of event times
- Within the compensator framework we characterize a class of cause and effect default models that are based on incomplete information
- These compensator models can be calibrated to the quality of information available to investors