

PRICING CREDIT FROM THE TOP DOWN WITH AFFINE POINT PROCESSES

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Abstract

A portfolio credit derivative is a contingent claim on the aggregate loss of a portfolio of credit sensitive securities. We develop an economically motivated and computationally tractable top down valuation framework in which portfolio loss follows an *affine point process*. The magnitude of each loss is random and defaults are governed by an intensity that is driven by affine jump diffusion risk factors. The portfolio loss itself is a risk factor so past defaults influence future loss dynamics. This enables the top down model to capture feedback from events and it introduces a dependence structure among default rates, recovery rates and interest rates. An affine point process supports semi-analytical transform based pricing and calibration. Hedging is facilitated by random thinning. We demonstrate the features of the top down framework in the context of CDS index and tranche swaps.

Key words: Self-affecting point process, affine point process, Hawkes process, portfolio credit derivative, index swap, tranche, transform, random thinning, pricing, calibration, hedging.

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1 Introduction

Defaults cluster. This is illustrated in Figure 1, which records the number of defaults among Moody's issuers by year. The clustering is driven by firm sensitivity to common economic factors, but it can also come from the feedback of an individual firm event to the aggregate level. A single default often results in the widening of credit spreads across the board. This is exemplified in Figure 2, which shows the impact of Delphi's default and GM's announcement of a huge quarterly loss on the spread of the Dow Jones CDX Crossover North America index swap.

In this article we develop a new *affine point process* framework to price and hedge a portfolio credit derivative such as the CDX index swap, an index tranche or an index option. In this top down approach, a credit derivative is a path-dependent contingent claim on a portfolio loss process, which is a point process that records financial loss due to default. We assume that the magnitude of each loss is random and that defaults are governed by an intensity that is a function of market wide risk factors. The loss process itself maybe a risk factor, in which case it is *self-affecting* since past defaults influence the future evolution of the portfolio loss. This captures the feedback effects of events seen in the index spread Figure 2. It also introduces a dependence structure among intensity, recovery rates and risk-free interest rates. The correlation of the intensity to the interest rate accounts for the flight-to-quality that often follows a series of defaults. The dependence of the intensity and the recovery rate addresses the negative relation between default and recovery rates observed in practice.

The affine point process framework is computationally tractable since the risk factors follow an affine jump diffusion. Based on the results of Duffie et al. (2000), we show that the transform of an affine point process is an exponentially affine function of the risk factors with coefficients that satisfy the Riccati ordinary differential equations. The transform determines the conditional distribution of future portfolio loss and the prices of contingent claims on the portfolio loss. This extends the reach of the transform methods pioneered by Heston (1993) for equity derivatives and leads to computationally efficient credit derivatives valuation, hedging and calibration. To demonstrate this, we implement these methods and provide numerical examples of model index and tranche spreads implied by the Hawkes process. The Hawkes process is a self-affecting affine point process whose intensity is driven by two random sources: the timing and recovery of past defaults. At a default, the intensity increases as a function of the realized loss. Between events, the intensity reverts to its long-run mean.

A portfolio credit derivative is sensitive to changes in the credit spread of the underlying names. The hedging of this exposure is important in practice. To price an instrument referenced on a constituent name and estimate hedge deltas, we use *random thinning* as proposed in Giesecke and Goldberg (2005). Here a fraction of the intensity is allocated to a given name. The resulting single name intensities reflect the dependence structure in the portfolio. The prices of single name instruments are exponentially affine functions of the

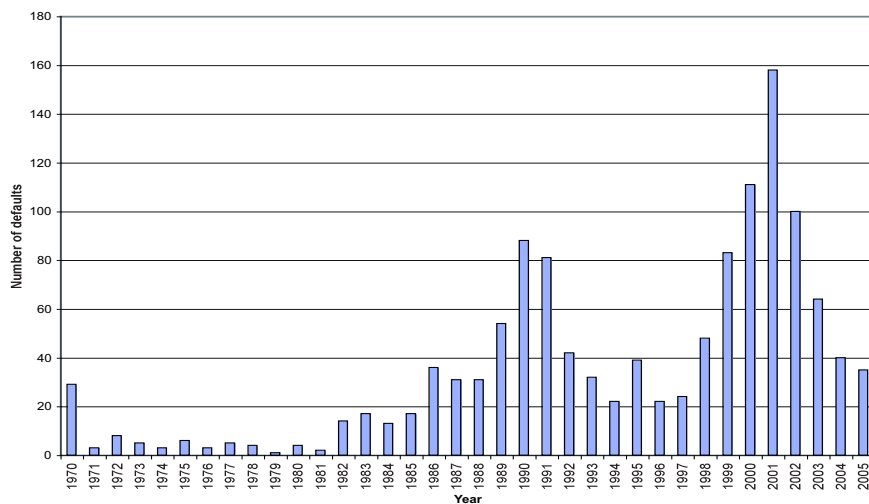


Figure 1: Number of Moody's rated issuers that defaulted in a given year. Source: Moody's.

risk factors with coefficients determined by Riccati-like ordinary differential equations. As above, this leads to computational tractability.

Model calibration is a two-step procedure. First an affine point process model is fit to standard index and tranche spreads. The resulting model can be used to consistently price tranches with non-standard attachment points and maturities, and derivatives that depend on the loss dynamics such as forward starting tranches and options on indexes and tranches. Subsequently, a parametric thinning function is calibrated to each single name swap curve. The parameters of the portfolio intensity are not affected so the fit to the multi-name market remains intact. The thinning function is used to estimate the delta hedge ratio for a portfolio constituent.

1.1 Related literature

Top down models fall into two categories, which are distinguished by their starting points. In the *forward approach*, which descends from the interest rate models in Heath et al. (1992), one defines a forward portfolio default rate and imposes conditions on its time evolution to guarantee the existence of a corresponding portfolio loss process. An input is the arbitrage-free initial loss distribution for all future times. From this initial state, the time evolution of the conditional loss distribution is simulated to estimate credit derivative prices. Examples of this approach include Schönbucher (2005), Sidenius et al. (2005) and Bennani (2005). These contributions are distinguished by their definitions of forward portfolio default rate.

We follow the *intensity approach* developed in Giesecke and Goldberg (2005). Here the portfolio loss process is specified in terms of a portfolio default rate and a distribution for loss given default. A precursor to this approach is in Davis and Lo (2001), who consider

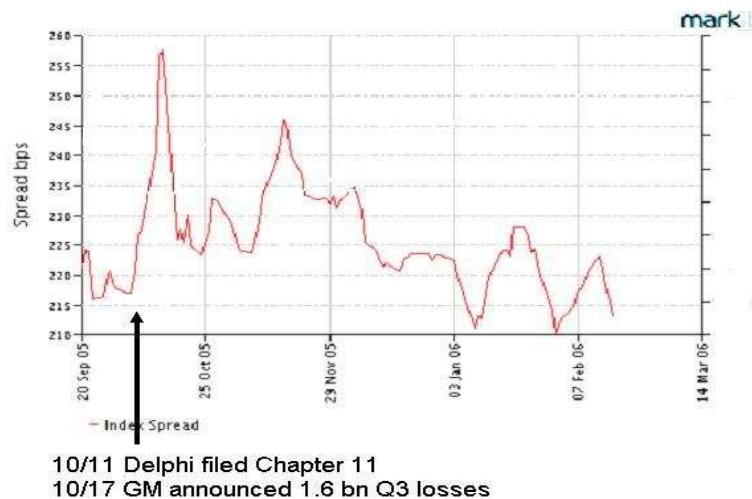


Figure 2: Dow Jones CDX.NA.XO 5-year credit swap index spread, which consists of 35 equally weighted credit default swaps referenced on BB and BBB rated issues. The time series shows the significant impact on the index spread of Delphi's default and GM's announcement of losses. Delphi and GM are not in the index. Source: Markit Partners.

a homogeneous portfolio in which an event ramps up the portfolio intensity by a fixed factor for an exponential time. Ordinary differential equations govern the distribution of the default process. A more recent example is Giesecke and Tomecek (2005), where stochastic time change techniques are used to produce both affine and non-affine self-affecting models of the portfolio loss process. In Longstaff and Rajan (2006) defaults are driven by three independent doubly stochastic processes that model idiosyncratic, sector specific and economy wide events. Brigo et al. (2006) model doubly stochastic event arrivals by a mixed compound Poisson process and its generalizations. Semi-analytic pricing is a common feature of these intensity based models so simulation is not required. Further, random thinning can be used to estimate hedge ratios in these models.

Some intensity based top down models have bottom up counterparts. In a bottom up model, the intensity of each portfolio constituent are the primitives. The dependence among firms must be built into the single name models. An example is a doubly stochastic setting where firm intensities are driven by common factors, as in Duffie and Garleanu (2001) and Mortensen (2005). Conditional on the factors, firm defaults are independent so the distribution of portfolio loss is easily calculated. It is challenging to relax the conditional independence and to incorporate the feedback effect of events. As shown in Jarrow and Yu (2001) and Frey and Backhaus (2004), computational tractability remains a major issue for portfolios with realistic features. While economically plausible, the frailty models of Collin-Dufresne et al. (2003), Duffie et al. (2006), Giesecke (2004) and Schönbucher (2004) focus on the marginal and joint distributions of the default times. Further steps

are needed to generate the portfolio loss process.

1.2 Structure of this article

In Section 2 we illustrate the top down approach to pricing credit in the context of index and tranche swaps. Appendix B contrasts this with an alternative bottom up approach. In Section 3 we consider the self-affecting Hawkes process as a basic model for portfolio defaults and losses. We give its conditional transform to support transform based pricing, and analyze the model index and tranche spreads implied by this model. We generalize in Section 4 by giving the definition and conditional transform of an affine point process. We also provide further parametric examples of affine point processes. Section 5 explains how to hedge single name exposure using random thinning.

2 CDS index contracts

We consider a portfolio of credit sensitive securities referenced on a collection of firms. The *default process* N counts defaults; N_t is the number of firms that have defaulted by time t . The *loss process* L records financial loss: L_t is the cumulative portfolio loss due to default at time t . These and all other processes appearing in this article are defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, Q)$ that satisfies the usual conditions. The probability measure Q is a pricing measure with respect to the interest rate $r > 0$.

A single name CDS allows investors to buy or sell default protection on an individual firm. We consider analogous *index contracts* that cover losses in a standardized index portfolio. Like a single name CDS, an index contract is structured as a swap consisting of two payment streams. Both default leg and premium leg depend on the cumulative portfolio losses. The premium rate equates the values of the two legs at contract inception.

2.1 Index swap

An index swap is based on an index portfolio whose n constituent single name swaps have a common notional I/n , common maturity date T and common premium payment dates (t_m) . The default leg of an index swap is a stream of payments that cover portfolio losses as they occur. Its value at time $t \leq T$ is given by the cumulative discounted losses

$$\begin{aligned} D_t &= E \left[\int_t^T e^{-r(s-t)} dL_s \mid \mathcal{G}_t \right] \\ &= e^{-r(T-t)} E[L_T \mid \mathcal{G}_t] - L_t + r \int_t^T e^{-r(s-t)} E[L_s \mid \mathcal{G}_t] ds. \end{aligned} \quad (1)$$

The premium leg of an index CDS consists of payments that are proportional to the

premium notional I_t , which is given by

$$I_t = I \left(1 - \frac{N_t}{n} \right). \quad (2)$$

The value of I_t is the total notional on the names that have survived until time t . Letting S denote the premium rate, the value of the premium leg is given by¹

$$\begin{aligned} P_t(S) &= E \left[\sum_{t_m \geq t} e^{-r(t_m-t)} S \bar{\alpha}_m I_{t_m} \mid \mathcal{G}_t \right] \\ &= SI \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m \left(1 - \frac{1}{n} E[N_{t_m} \mid \mathcal{G}_t] \right) \end{aligned} \quad (3)$$

where $\bar{\alpha}_m$ is the appropriate day count fraction for the period m . A top down estimate of the *CDS index value* S_t at time t is the solution $S = S_t$ to the equation $D_t = P_t(S)$. Using equations (1) and (3),

$$S_t = \frac{e^{-r(T-t)} E[L_T \mid \mathcal{G}_t] - L_t + r \int_t^T e^{-r(s-t)} E[L_s \mid \mathcal{G}_t] ds}{I \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m \left(1 - \frac{1}{n} E[N_{t_m} \mid \mathcal{G}_t] \right)}. \quad (4)$$

Formula (4) highlights the fact that the index value S_t depends only on conditional expected portfolio losses and defaults at future dates. In Appendix B we discuss an equivalent bottom up calculation of S_t as a weighted average of the spreads of the portfolio constituents. In the special case where the index portfolio consists of a single name only, formula (4) gives the premium on a single name swap referenced on that firm.

2.2 Tranche swap

Investors interested in narrower risk profiles invest in contracts based on a tranche or slice of the index specified by a lower attachment point $\underline{K} \in [0, 1]$ and an upper attachment point $\bar{K} \in (\underline{K}, 1]$. The product of the difference $K = \bar{K} - \underline{K}$ and the contract notional I is the tranche notional.

The default leg of a tranche swap is a stream of payments that cover portfolio losses as they occur, given that the cumulative losses are larger than \underline{K} but do not exceed \bar{K} . The cumulative payments at time t , denoted U_t , are

$$U_t = (L_t - I\underline{K})^+ - (L_t - I\bar{K})^+. \quad (5)$$

Note that $U_t \in [0, IK]$ almost surely. The value of these payments at time $t \leq T$ is

$$\begin{aligned} D_t &= E \left[\int_t^T e^{-r(s-t)} dU_s \mid \mathcal{G}_t \right] \\ &= e^{-r(T-t)} E[U_T \mid \mathcal{G}_t] - U_t + r \int_t^T e^{-r(s-t)} E[U_s \mid \mathcal{G}_t] ds. \end{aligned} \quad (6)$$

¹Here and elsewhere in this article, we neglect the accrual term.

Formula (6) is analogous to formula (1) for the value of an index swap default leg. In particular, the latter can be viewed as the default leg of a tranche swap for which $\underline{K} = 0$ and $\overline{K} = 1$.

The premium leg of a tranche swap has two parts. The first part is an upfront payment, which is expressed as a fraction F of the tranche notional KI . The second part consists of payments that are proportional to the premium notional, described by the process I given by²

$$I_t = KI - U_t. \quad (7)$$

The value of this process is the difference between the tranche notional and the cumulative default losses. Note the difference between the tranche premium notional defined in formula (7) and the index premium notional defined in formula (2). The latter does not take account of loss given default. Letting S denote the premium rate, the value of the premium leg is given by

$$\begin{aligned} P_t(F, S) &= FKI + E \left[\sum_{t_m \geq t} e^{-r(t_m-t)} S \bar{\alpha}_m I_{t_m} \mid \mathcal{G}_t \right] \\ &= FKI + S \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m (KI - E[U_{t_m} \mid \mathcal{G}_t]). \end{aligned} \quad (8)$$

Setting $D_t = P_t(F, S)$ for a fixed upfront payment rate F gives a top down value $S = S_t$ for the time t premium rate S_t on the tranche swap,

$$S_t = \frac{e^{-r(T-t)} E[U_T \mid \mathcal{G}_t] - U_t + r \int_t^T e^{-r(s-t)} E[U_s \mid \mathcal{G}_t] ds - FKI}{\sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m (KI - E[U_{t_m} \mid \mathcal{G}_t])}. \quad (9)$$

Similarly, setting $D_t = P_t(F, S)$ for a fixed premium rate S gives a top down value $F = F_t$ for the time t upfront rate F_t ,

$$\begin{aligned} F_t &= \frac{1}{KI} \left(e^{-r(T-t)} E[U_T \mid \mathcal{G}_t] - U_t + r \int_t^T e^{-r(s-t)} E[U_s \mid \mathcal{G}_t] ds \right. \\ &\quad \left. - S \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m (KI - E[U_{t_m} \mid \mathcal{G}_t]) \right). \end{aligned} \quad (10)$$

Formulae (9) and (10) show that the premium and upfront rates depend only on the value $e^{-r(s-t)} E[U_s \mid \mathcal{G}_t]$ of *call spreads* on the portfolio loss process L with strikes \underline{K} and \overline{K} and maturities $s \in (t, T]$. Note that in case the portfolio consists of a single name only, the premium on the tranche with $\underline{K} = 0$ and $\overline{K} = 1$ prescribed by formula (9) coincides with the single name premium only if there is zero recovery at default.

²The definition of the premium notional for the super senior tranche is slightly different. It is given by $KI - U_t - (\frac{I}{n} N_t - L_t)$.

3 The Hawkes process as a model for portfolio loss

As illustrated above, many portfolio credit derivatives are path dependent contingent claims on N or L . To value these derivatives, we need models for the dynamics of N and L that are economically reasonable and computationally tractable. Giesecke and Goldberg (2005) propose a *self-affecting point process* whose intensity jumps at default times. Self-affecting point processes capture the feedback effects of events. We show that they are also computationally tractable. We begin with an example and generalize in Section 4.

3.1 Specification

Suppose defaults arrive with intensity λ given by

$$\lambda_t = c(t) + \int_0^t d(t-s) dL_s. \quad (11)$$

The first to default intensity c is a nonnegative deterministic function of time, $c(0) > 0$ and the impact of a loss on the intensity is governed by the function

$$d(s) = \delta e^{-\kappa s}, \quad s \geq 0, \quad (12)$$

with $\kappa \geq 0$ and $\delta \geq 0$. The intensity is driven by two factors: the timing of past events and the recovery at these events. At default, the intensity increases by the realized loss scaled by the sensitivity parameter δ . A higher rate of recovery leads to a smaller jump of the intensity, so that default and recovery rates are negatively correlated. The impact of an event decays exponentially over time with rate κ . A sample intensity path is in Figure 3.

The two dimensional process $J = (L, N)^\top$ is a Hawkes (1971) process whose components share common event times. Since its intensity increases at an event, the Hawkes process is positively self-affecting, or *self-exciting*. If $\kappa = 0$, then J is a birth process. If $\delta = 0$, then J is a time inhomogeneous compound Poisson process.

The intensity has dynamics given by

$$d\lambda_t = \kappa(\varrho(t) - \lambda_t) dt + \delta dL_t \quad (13)$$

where $\varrho(t)$ is a deterministic function describing the level of mean reversion. It satisfies

$$c(t) = c(0)e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-s)} \varrho(s) ds. \quad (14)$$

If $\varrho(t)$ is a positive constant λ_∞ , formula (14) reduces to

$$c(t) = \lambda_\infty + e^{-\kappa t}(c(0) - \lambda_\infty)$$

and $\lambda_\infty = \lim_{t \rightarrow \infty} c(t)$.

Figure 4 plots a histogram of 500 simulated event arrivals of the Poisson and Hawkes processes. The Hawkes process is based on the parameters $\kappa = \delta = 5$ and $\varrho(t) = \lambda_\infty = c(0) = 0.7$. The Poisson process is the special Hawkes process for which $\kappa = \delta = 0$ so that $\lambda_t = \lambda_\infty$. While the Poisson arrivals are evenly distributed over time due to the order statistics property, the Hawkes arrivals are *clustered* thanks to the self-affecting property.

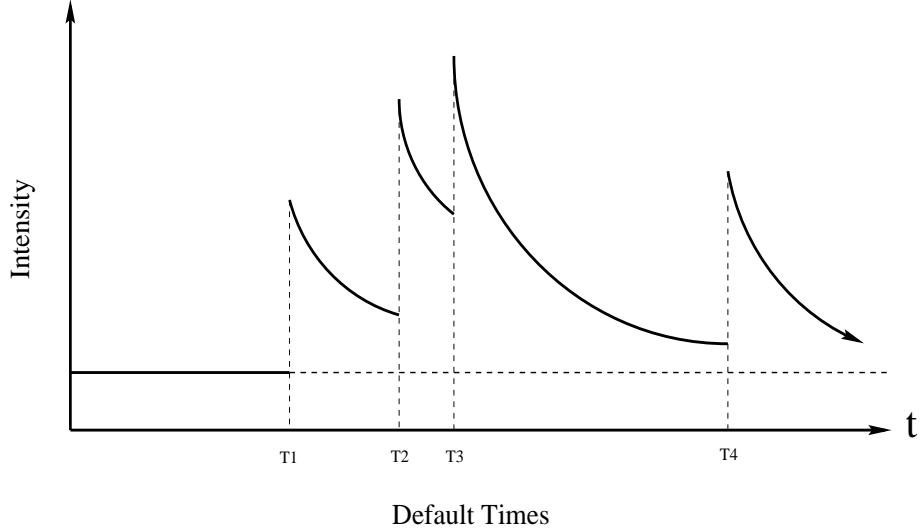


Figure 3: Intensity path of the Hawkes process for which the first-to-default intensity $c(t)$ is constant. A jump in the intensity represents the impact of an event.

3.2 Conditional transform

We give the conditional transform of the point process J . It can be inverted to obtain the conditional distribution of L and N . This result is a special case of Proposition 4.4 below, and can also be obtained as the solution to partial integro-differential equation as shown in Appendix A. Throughout, we denote by \mathbb{C}^n the set of n -tuples of complex numbers and by \mathbb{C}_-^n the set of n -tuples of complex numbers with non-positive real parts.

Proposition 3.1. *Suppose that $\varrho(t) = \lambda_\infty$ and that the loss at default is independent of N and has distribution ν on $(0, \infty)$. The conditional transform of J is given by*

$$E[e^{u \cdot J_s} | \mathcal{G}_t] = e^{a(u,t,s) + b(u,t,s)\lambda_t + u \cdot J_t} \quad (15)$$

where $t \leq s$, $u \in \mathbb{C}_-^2$ and the coefficient functions $a(t) = a(u, t, s)$ and $b(t) = b(u, t, s)$ satisfy the following ordinary differential equations:

$$\partial_t b(t) = \kappa b(t) - \varphi(\delta b(t), u) + 1 \quad (16)$$

$$\partial_t a(t) = -\kappa \lambda_\infty b(t) \quad (17)$$

with boundary conditions $a(s) = b(s) = 0$, where φ is the jump transform

$$\varphi(c, u) = e^{u \cdot (0,1)^\top} \int_{\mathbb{R}_+} e^{(c+u \cdot (1,0)^\top)z} d\nu(z).$$

In some cases the ODEs can be solved analytically. For example, suppose that $\kappa = 0$ so that J is the classical birth process. Suppose further that the loss at default is equal

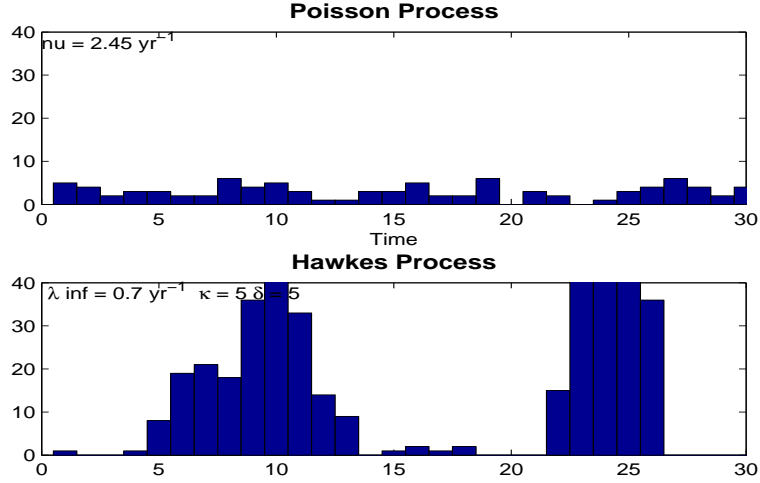


Figure 4: Histogram of 500 simulated event arrivals of the Poisson and Hawkes processes that shows the clustering of events in the Hawkes case. The Hawkes process is based on the parameters $\kappa = \delta = 5$ and $\varrho(t) = \lambda_\infty = c(0) = 0.7$. The Poisson process is the special Hawkes process for which $\kappa = \delta = 0$ so that $\lambda_t = \lambda_\infty = c(0)$. We choose $c(0) = 2.45$ to match the expected number of Hawkes events over 30 years.

to 1 so that $L = N$. Then the transform of J is determined by the coefficient functions

$$b(u, t, s) = -\frac{1}{\delta} \log \left(e^{u \cdot (1,1)^\top} + e^{\delta(s-t)} (1 - e^{u \cdot (1,1)^\top}) \right)$$

$$a(u, t, s) = 0.$$

In the general case, the ODEs are quickly solved numerically, using the Runge-Kutta algorithm for example. The corresponding transform can be easily inverted numerically using Fast Fourier Transform (FFT) techniques.

Figure 5 shows the distribution of the number of defaults in one year for the birth and Hawkes processes. For the birth process ($\kappa = 0$) we fix the initial intensity and vary the impact parameter δ . The higher δ , the more sensitive is the intensity to an event. For the Hawkes process we fix the initial intensity and the impact sensitivity and vary the mean reversion rate κ . The higher κ , the faster decays the impact of an event on the intensity. Similar effects are shown on Figure 6, which plots the density of portfolio losses in one year. The distribution ν governing the loss at a default puts equal probability on the amounts $\{0.4, 0.6, 0.8, 1\}$.

Based on the conditional characteristic function, below we analyze the model spreads for index and tranche swaps implied by the Hawkes process with constant level of mean reversion $\varrho(t) = \lambda_\infty$ and jump size that is independent of the arrivals.

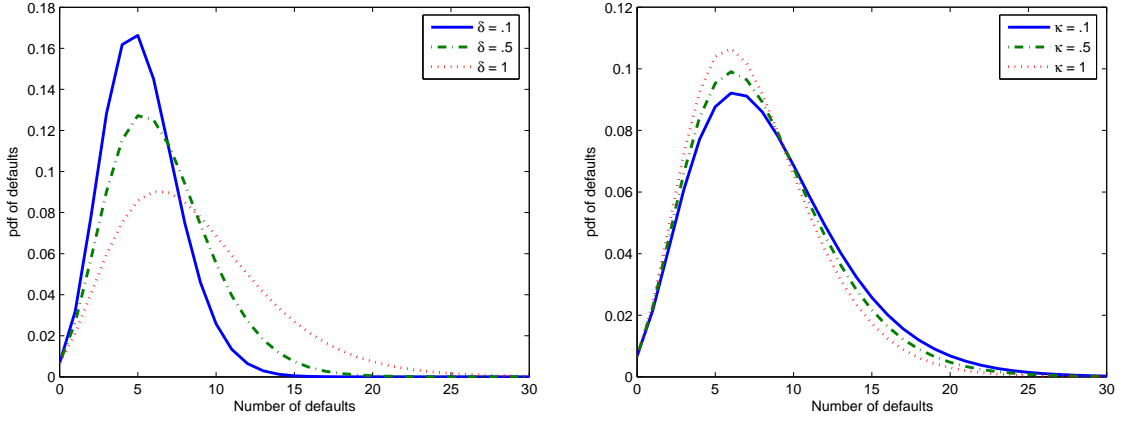


Figure 5: Distribution of the number of defaults in one year for the birth and Hawkes processes. For the birth process ($\kappa = 0$, left) we take $c(0) = 5$ and vary δ . For the Hawkes process (right) we take $c(0) = \varrho(t) = \lambda_\infty = 5$, $\delta = 1$ and vary κ .

3.3 Implied index spreads

We can calculate the model index spread S by formula (4), which expresses S in terms of the conditional expected loss $E[L_s | \mathcal{G}_t]$ and default count $E[N_s | \mathcal{G}_t]$ in case the jump distribution ν has finite expectation. We obtain these quantities by differentiating the conditional transform of J with respect to u and evaluating the derivative at $u = \mathbf{0} = (0, 0)^\top$. We multiply the result by the vector $(1, 0)^\top$ and $(0, 1)^\top$ to obtain $E[L_s | \mathcal{G}_t]$ and $E[N_s | \mathcal{G}_t]$, respectively. It is convenient to express the result in terms of new ODEs that can be solved directly. For $t \leq s$ we get

$$E[y \cdot J_s | \mathcal{G}_t] = (A(\mathbf{0}, t, s) + B(\mathbf{0}, t, s)\lambda_t + y \cdot J_t)e^{a(\mathbf{0}, t, s) + b(\mathbf{0}, t, s)\lambda_t}$$

where the functions $a(t)$ and $b(t)$ satisfy the ODEs (16) and (17) above and the functions $A(t) = A(u, t, s)$ and $B(t) = B(u, t, s)$ satisfy the following ODEs:

$$\partial_t B(t) = \kappa B(t) - (\delta B(t) + y \cdot (1, 0)^\top) \theta'(\delta b(t)) - y \cdot (0, 1)^\top \theta(\delta b(t)) \quad (18)$$

$$\partial_t A(t) = -\kappa \lambda_\infty B(t) \quad (19)$$

with boundary conditions $A(s) = B(s) = 0$, where

$$\theta(c) = \int_{\mathbb{R}_+} e^{cz} d\nu(z), \quad \theta'(c) = \int_{\mathbb{R}_+} z e^{cz} d\nu(z).$$

Figure 7 shows plots of the annualized index spread on a 5 year index as a function of the mean-reversion rate κ and the long-run mean λ_∞ . The higher κ , the faster decays the impact of an event and the lower the risk of large losses. The sensitivity of the spread to κ increases with λ_∞ . The spread is highly sensitive to changes in λ_∞ , which represents

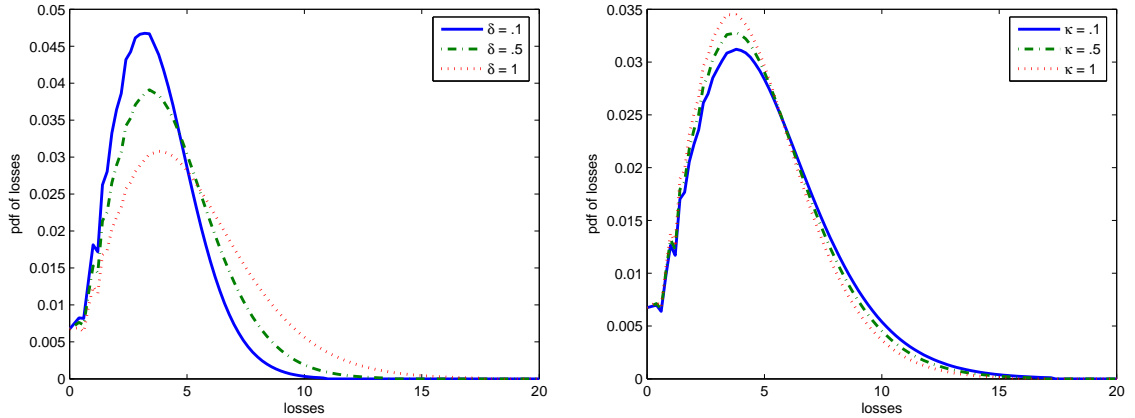


Figure 6: Density of the portfolio loss in one year for the birth and Hawkes processes. For the birth process ($\kappa = 0$, left) we take $c(0) = 5$ and vary δ . For the Hawkes process (right) we take $c(0) = \varrho(t) = \lambda_\infty = 5$, $\delta = 1$ and vary κ . The distribution ν governing the loss at a default puts equal probability on the amounts $\{0.4, 0.6, 0.8, 1\}$.

the mean index spread in the long run. A shift in λ_∞ can be interpreted as a shift in the spreads on the constituent names across the board. The sensitivity increases with κ .

Implicit in these calculations is the assumption that the underlying portfolio can, in theory, experience infinitely many defaults over an infinite time horizon. This assumption is innocuous in practice, since the index portfolio is adjusted frequently and defaulted names are substituted with new names. In effect there is an unlimited supply of names that might default at some point. Further, the probability of k defaults decays very rapidly as a function of k . Therefore, the law of the process N stopped at the n th arrival is very well approximated by the distribution of N if the number of names n is sufficiently large. If the portfolio is relatively small, then the distribution of the stopped process can be obtained from that of N by adjusting the probability of the n th bin, by adding the mass to the right of that bin. The probabilities of the bins to the left remain unchanged. For an example, see Definition 4.8 in Giesecke and Goldberg (2005) or Brigo et al. (2006).

3.4 Implied tranche spreads

The model tranche spread is calculated in terms of the value of options on the loss L with various maturities, see formula (9). The conditional density $f_t(x, s)$ at time t of the loss L_s is obtained by inverting the Fourier transform (15) and we calculate

$$E[(L_s - c)^+ | \mathcal{G}_t] = \int_c^\infty (x - c) f_t(x, s) dx$$

which is quickly and accurately evaluated numerically. Alternatively, we can directly invert the conditional transform of the option price, which is a simple function of the transform

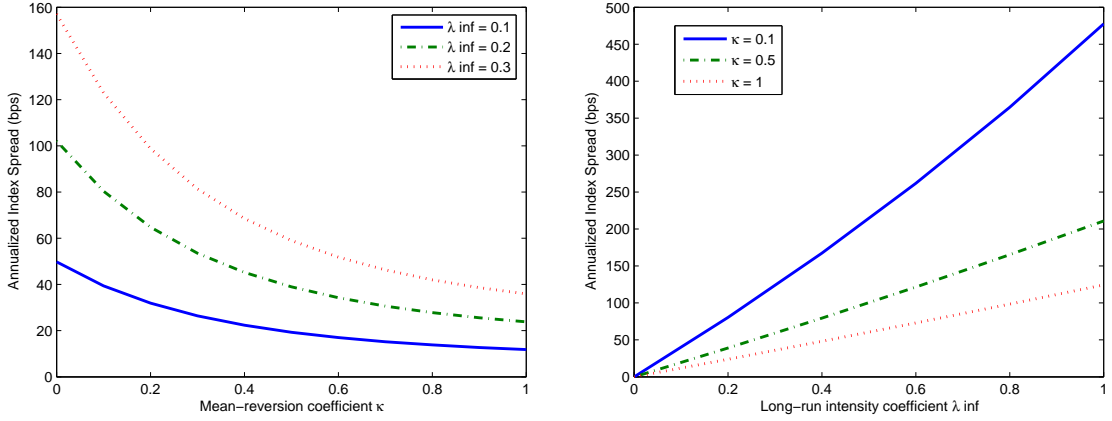


Figure 7: Annualized index spread for a five year maturity index as a function of the mean-reversion rate κ and the long-run mean λ_∞ . We set $c(0) = \varrho(t) = \lambda_\infty$ and $\delta = 1$. The distribution ν governing the loss at a default puts equal probability on the amounts $\{0.4, 0.6, 0.8, 1\}$. The risk-free rate $r = 5\%$ and the index notional $I = 125$.

(15), see Lee (2004). Another possibility is to decompose the option value into two parts, the expected values of $L_s 1_{\{L_s \geq c\}}$ and $c 1_{\{L_s \geq c\}}$. The transform of the first part is given in terms of ODEs similar to those derived above, see Duffie et al. (2000). We implemented all three alternatives and found that the first method is the most efficient.

In Figure 8 we plot the 5 year tranche spread as a function of the mean-reversion rate κ and the long-run mean intensity λ_∞ . Equity spreads exceed mezzanine and senior spreads since the equity tranche carries the highest risk. The higher κ , the faster the impact of a default on the portfolio decays, the lighter the tail of the loss distribution and the higher are equity spreads. Correspondingly, super senior spreads decrease with κ . Mezzanine spreads are less sensitive to κ . They decrease initially, but are roughly invariant or even slightly increase after a threshold.

An increase in λ_∞ represents a broad spread widening. Equity spreads increase sharply since the equity tranche absorbs losses right from inception. Mezzanine spreads increase only slightly while super senior spreads even decrease.

In Figure 9 we plot the 5 year tranche spread as a function of the impact sensitivity parameter δ . With an increase in δ mass is shifted in the tail of the loss distribution, so equity spreads decrease and super senior spreads increase. The effect on Mezzanine spreads is ambiguous.

In Figure 10 we plot the 5 year mezzanine and super senior spreads as a function of the volatility of the loss given default. Our numerical investigation showed that the equity tranche is almost insensitive to changes in the volatility. The sensitivity increases with the tranche seniority, as shown in the graph.

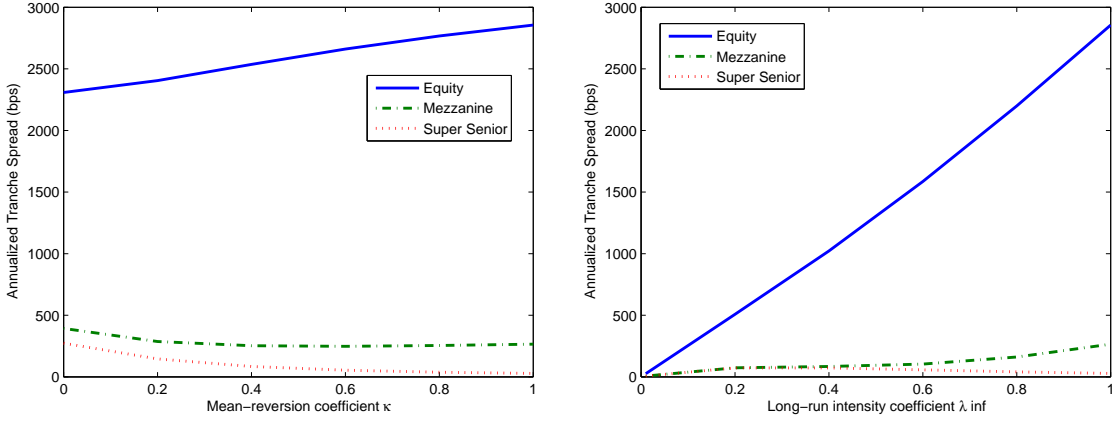


Figure 8: Annualized spread for five year maturity tranches as a function of the mean-reversion rate κ and the long-run mean λ_∞ . We set $c(0) = \varrho(t) = \lambda_\infty$ and $\delta = 0.5$. For the left panel $\lambda_\infty = 1$ and for the right panel $\kappa = 1$. The distribution ν governing the loss at a default puts equal probability on the amounts $\{0.4, 0.6, 0.8, 1\}$. The risk-free rate $r = 5\%$ and the index notional $I = 125$. The attachment points are $[0, 3\%]$ for Equity, $[3\%, 7\%]$ for Mezzanine and $[30\%, 100\%]$ for Super Senior.

4 Affine point processes

The Hawkes model is an example of an affine point process, which is a computationally tractable, intensity based model for portfolio default and loss. An affine point process has an intensity that is driven by an affine jump diffusion.

4.1 Definition

Consider a vector-valued point process J whose component processes share common event times that arrive with intensity λ . At each event time the jump size vector is drawn from a fixed distribution ν on \mathbb{R}_+^d that has mass zero at 0. The process J is an *affine point process* if for some functions $\Lambda_0(t) \in \mathbb{R}_+$ and $\Lambda_1(t) \in \mathbb{R}_+^d$ the intensity is given by

$$\lambda_t = \lambda(X_t, t) = \Lambda_0(t) + \Lambda_1(t) \cdot X_t,$$

where X is a strong solution to the stochastic differential equation

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t + dZ_t + \zeta dJ_t, \quad X_0 \in \mathbb{R}^d \quad (20)$$

where W is an \mathbb{R}^d -valued standard Brownian motion, $\mu(X)$ is the drift, $\sigma(X)$ is the volatility and Z is a sum of m \mathbb{R}_+^d -valued point processes. The component processes of a vector Z^i share common event times that arrive with intensity $h^i(X)$. The jump sizes are drawn from a distribution ν^i on \mathbb{R}_+^d that has mass zero at 0. The parameter ζ is a

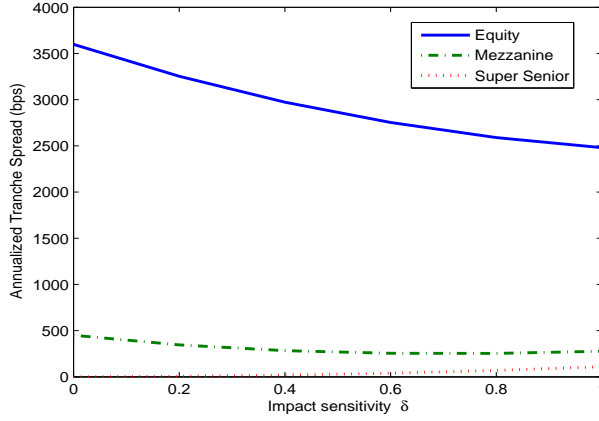


Figure 9: Annualized spread for five year maturity tranches as a function of the impact sensitivity parameter δ . We set $c(0) = \varrho(t) = \lambda_\infty = 1$ and $\kappa = 1$. The distribution ν governing the loss at a default puts equal probability on the amounts $\{0.4, 0.6, 0.8, 1\}$. The risk-free rate $r = 5\%$ and the index notional $I = 125$. The attachment points are $[0, 3\%]$ for Equity, $[3\%, 7\%]$ for Mezzanine and $[30\%, 100\%]$ for Super Senior.

d -dimensional diagonal matrix. The coefficient functions have affine dependence on X :

$$\begin{aligned} \mu(x) &= K_0 + K_1 \cdot x, \quad K_0 \in \mathbb{R}^d, \quad K_1 \in \mathbb{R}^{d \times d} \\ (\sigma(x)\sigma(x)^\top)_{jk} &= (H_0)_{jk} + (H_1)_{jk} \cdot x, \quad H_0 \in \mathbb{R}^{d \times d}, \quad H_1 \in \mathbb{R}^{d \times d \times d} \\ h^i(x) &= M_0^i + M_1^i \cdot x, \quad M_0^i \in \mathbb{R}, \quad M_1^i \in \mathbb{R}^d, \quad i = 1, 2, \dots, m. \end{aligned}$$

If J is an affine point process driven by X , then $(J, X)^\top$ is a Markov process whose transition semi-group has an infinitesimal generator A , which is defined on a bounded C^2 function g with bounded first and second derivatives by

$$\begin{aligned} Ag(t, x, j) &= \partial_x g(t, x, j) \cdot \mu(x) + \frac{1}{2} \text{tr} \left(\partial_{xx} g(t, x, j) \sigma(x) \sigma(x)^\top \right) \\ &+ \sum_i h^i(x) \int_{\mathbb{R}_+^d} [g(t, x + y, j) - g(t, x, j)] d\nu^i(y) \\ &+ \lambda(x, t) \int_{\mathbb{R}_+^d} [g(t, x + \zeta y, j + y) - g(t, x, j)] d\nu(y). \end{aligned} \quad (21)$$

A natural application of an affine point process J is to model the dynamics of defaults and losses in a portfolio of credit sensitive securities. The event times of J are governed by an intensity λ , which is an affine function of a vector X of risk factors that drive the credit risk in the economy.

If J itself is a risk factor, then at least one diagonal entry of ζ is non-zero. It follows that J is self-affecting: the path of J observed up to a given time influences the future

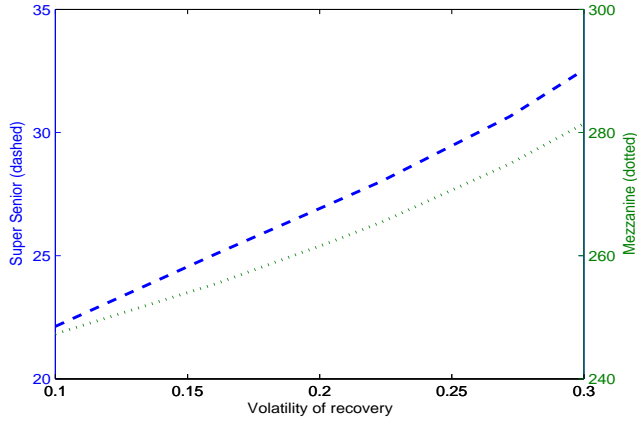


Figure 10: Annualized spread for five year maturity tranches as a function of the impact of the volatility of the recovery. We set $c(0) = \varrho(t) = \lambda_\infty = 1$, $\delta = 0.5$ and $\kappa = 1$. The risk-free rate $r = 5\%$ and the index notional $I = 125$. The attachment points are $[3\%, 7\%]$ for Mezzanine and $[30\%, 100\%]$ for Super Senior.

evolution of J . The Hawkes process described in Section 3.1 is an example of a self-affecting affine point process. An important generalization is made by introducing Brownian terms and independent jump terms in the intensity; see Section 4.2 for examples. The Brownian terms model the diffusive fluctuation in the portfolio intensity and can be driven by a stochastic volatility. The jump terms model the sensitivity of the intensity to market events, such as macro-economic shocks or defaults to names that are outside the portfolio.

In case ζ is a matrix of zeroes, J is not a risk factor so its intensity is independent of the events. A doubly stochastic process, in which J is a time-inhomogeneous Poisson process conditional on the path of the risk factors,³ is an example of an affine point process that is not self-affecting.

4.2 Self-affecting examples

We give examples of two dimensional self-affecting affine point processes $J = (L, N)^\top$. In each case, the risk factors take the form $X = (\lambda, N)^\top$ so that $\Lambda_0 = 0$ and $\Lambda_1 = (1, 0)^\top$. Further, the intensity λ takes the general form

$$\lambda_t = c_t + \delta \int_0^t e^{-\kappa(t-s)} dL_s \quad (22)$$

where the non-negative process c is the first-to-default intensity with initial value $c_0 = \lambda_0 > 0$. It follows that $X_0 = (c_0, 0)^\top$ and the diagonal of ζ is equal to $(\delta, 1)^\top$ where the impact parameter $\delta > 0$ is the sensitivity of the intensity to a loss. The constant κ is

³See, for example Giesecke and Goldberg (2005, Definition 4.8).

non-negative and if $\kappa > 0$, it is the rate at which the intensity reverts to a constant target level, denoted λ_∞ .

Example 4.1. Consider the prototypical Hawkes process featured in Section 3. Its first-to-default intensity is the deterministic process

$$c(t) = \lambda_\infty + (c(0) - \lambda_\infty)e^{-\kappa t}.$$

Together with expression (22), this implies that λ satisfies

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t) dt + \delta dL_t.$$

Consequently, H_0 is a matrix of zeros, H_1 is a tensor of zeros and

$$K_0 = \begin{pmatrix} \kappa\lambda_\infty \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix}. \quad (23)$$

Example 4.2. Consider the affine point process J whose first-to-default intensity is the Gaussian diffusion process

$$c_t = \lambda_\infty + (c_0 - \lambda_\infty)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s.$$

The intensity satisfies

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)dt + \sigma dW_t + \delta dL_t. \quad (24)$$

Between events, the intensity drifts stochastically towards λ_∞ with fluctuations driven by the Brownian motion W . It is generated by setting H_1 to be a tensor of zeroes and

$$K_0 = \begin{pmatrix} \kappa\lambda_\infty \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (25)$$

Note that this process can be negative with positive probability.

Example 4.3. Consider the affine point process J whose first-to-default intensity is the diffusion process

$$c_t = \lambda_\infty + (c_0 - \lambda_\infty)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} \sqrt{\lambda_s} dW_s. \quad (26)$$

so that the intensity satisfies⁴

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t) dt + \sigma\sqrt{\lambda_t} dW_t + \delta dL_t. \quad (27)$$

Between events, the intensity drifts stochastically toward λ_∞ with fluctuations driven by W . It is generated by setting H_0 to be a matrix of zeroes and

$$K_0 = \begin{pmatrix} \kappa\lambda_\infty \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix}, \quad (H_1)_{111} = \sigma^2. \quad (28)$$

⁴The coefficient $\sqrt{\lambda_t}$ is not predictable. Nevertheless, it can be integrated against the Brownian motion W since it is progressively measurable, see for example Revuz and Yor (2005, Chapter I, Proposition 4.8 and Chapter IV).

Further examples include point processes whose intensities are subject to independent jumps, which model the impact of market events other than defaults. Another class of models allows for a stochastic volatility driving the diffusion component of the intensity.

4.3 Conditional transform

An affine point process generates pricing models that are computationally tractable. This is because the vector of risk factors X generating an affine point process J is an affine jump diffusion process in the sense of Duffie et al. (2000). Consider the “discounted” conditional transform of the affine point process J ,

$$\psi(u, X_t, J_t, t, s) = E[e^{-\int_t^s \rho(X_v, v) dv} e^{u \cdot J_s} | \mathcal{G}_t] \quad (29)$$

where ρ is a non-negative process with affine dependence on X :

$$\rho(x, t) = R_0(t) + R_1(t) \cdot x, \quad R_0(t) \in \mathbb{R}, \quad R_1(t) \in \mathbb{R}^d.$$

The process ρ can model the extended intensity of a portfolio constituent as discussed below in Section 5, or the stochastic interest rate. Under technical conditions stated in Proposition 4.4 below, the discounted conditional transform (29) is given by

$$\psi(u, X_t, J_t, t, s) = e^{\alpha(u, t, s) + \beta(u, t, s) \cdot X_t + u \cdot J_t} \quad (30)$$

where $u \in \mathbb{C}_-^d$, $t \leq s$ and the coefficient functions $\alpha(t) = \alpha(u, t, s)$ and $\beta(t) = \beta(u, t, s)$ satisfy the ordinary differential equations

$$\begin{aligned} \partial_t \beta(t) = & R_1(t) - K_1^\top \beta(t) - \frac{1}{2} \beta(t)^\top H_1 \beta(t) - \sum_{i=1}^m M_1^i(\theta^i(\beta(t)) - 1) \\ & - \Lambda_1(t)(\theta(\zeta \beta(t) + u) - 1) \end{aligned} \quad (31)$$

$$\begin{aligned} \partial_t \alpha(t) = & R_0(t) - K_0 \cdot \beta(t) - \frac{1}{2} \beta(t)^\top H_0 \beta(t) - \sum_{i=1}^m M_0^i(\theta^i(\beta(t)) - 1) \\ & - \Lambda_0(t)(\theta(\zeta \beta(t) + u) - 1) \end{aligned} \quad (32)$$

with boundary conditions $\alpha(s) = 0$ and $\beta(s) = \mathbf{0}$ and jump transforms

$$\theta^i(c) = \int_{\mathbb{R}_+^d} e^{c \cdot z} d\nu^i(z) \quad \text{and} \quad \theta(c) = \int_{\mathbb{R}_+^d} e^{c \cdot z} d\nu(z).$$

Proposition 4.4. *The discounted conditional transform of an affine point process J driven by the affine jump diffusion X is given by*

$$E[e^{-\int_t^s \rho(X_v, v) dv} e^{u \cdot J_s} | \mathcal{G}_t] = e^{\alpha(u, t, s) + \beta(u, t, s) \cdot X_t + u \cdot J_t}$$

if the coefficient functions α and β uniquely solve the differential equations (31) and (32) and if for $(u, s) \in \mathbb{C}^d \times [0, \infty)$ the following expectations are all finite:

$$(1) E[|\Psi(u, X_s, J_s, 0, s)|]$$

$$\Psi(u, X_t, J_t, t, s) = e^{-\int_0^t \rho(X_v, v) dv} \psi(u, X_t, J_t, t, s)$$

$$(2) E[(\int_0^s |\eta(t)|^2 dt)^{1/2}]$$

$$\eta(t) = \Psi(u, X_t, J_t, t, s) \beta(t)^\top \sigma(X_t)$$

$$(3) E[\int_0^s |\gamma(t)| dt]$$

$$\gamma(t) = \Psi(u, X_t, J_t, t, s) (\sum_{i=1}^m (\theta^i(\beta(t)) - 1) h^i(X_t) + (\theta(\zeta\beta(t) + u) - 1) \lambda(X_t, t))$$

We prove this statement in Appendix A by following an argument of Duffie et al. (2000). There are two alternative approaches. One approach is based on the infinitesimal generator. Here, the relevant technical assumptions are that the affine point process be stochastically continuous and regular as defined in Duffie et al. (2003). This path leads to a partial integro-differential equation (PIDE) that can be transformed into the Riccati equations by a change of variable. We give an argument of this type for the Hawkes process in Appendix A. An alternative approach is based on the stochastic time-change techniques developed in Giesecke and Tomecek (2005). Here, the affine point process is considered as a transformed Poisson process.

Inversion of the transform given in Proposition 4.4 yields the conditional distribution of an affine point process. Under technical assumptions stated in Proposition A.1 in Appendix A, the transform can be differentiated to obtain the conditional expected defaults and losses required to give an affine point process value for the index and tranche spreads in formula (4) and (9).

5 Hedging single names

The parameters of an affine point process model for loss and default can be backed out from market index and tranche spreads by generalized least squares or another standard fitting method using formulas (4) and (9). Compatibly calibrated models for the constituent reference names of a portfolio are required to *hedge* the exposure to single name spread changes and defaults. To determine the amount of protection to be bought or sold, we require the sensitivity of the mark-to-market of a position to a change in the swap curve of a constituent name.

5.1 Random thinning

We consistently generate single name models from a portfolio model using a version of *random thinning* proposed in Giesecke and Goldberg (2005). Consider a portfolio of n

names indexed by $i = 1, 2, \dots, n$. Suppose the default counting process N is of the form

$$N = \sum_{i=1}^n N^i \quad (33)$$

where each N^i is an \mathbb{R}_+ -valued non-explosive counting process with intensity

$$\lambda^i = Y^i \lambda. \quad (34)$$

Here, λ is the intensity of N and the Y^i are nonnegative stochastic processes that sum to 1. The thinning variable Y_t^i allocates a fraction of the portfolio intensity λ_t to name i . It can be interpreted as the conditional probability that name i experiences an event, given there is an event in the next instant. An alternative interpretation of the thinning process in terms of spread returns is in Appendix C. Note that the λ^i sum to the portfolio intensity λ as required by formula (33).

Firm i defaults at the first jump time of the counting process N^i .⁵ The corresponding marginal conditional survival probability at time t is given by

$$p_t^i(s) = P[N_s^i = 0 \mid \mathcal{G}_t] = E[e^{-\int_t^s \lambda_v^i dv} \mid \mathcal{G}_t] \quad (35)$$

and depends on the parameters of the models chosen for λ and Y^i . Formula (35) can be further generalized to value credit derivatives referenced on firm i , see Duffie et al. (1996). These formulae are used to calibrate a parametric thinning vector (Y^1, \dots, Y^n) from the securities referenced on the constituent names. For example, the parameters of each Y^i can be chosen to match the survival probability functions (35) implied by the single name swap curves observed in the market, subject to the nonnegativity and summation constraints. Note that the fit of the portfolio intensity model λ to the multi-name market is not affected.

5.2 Hedging with an affine point process model

We outline a procedure to estimate to first order the derivative Δ^i of the mark-to-market of the multi-name position with respect to a constituent name mark-to-market. The product of Δ^i with the corresponding notional is the amount of protection on name i required to hedge the change in the multi name position to changes in the market spread of name i .

Suppose the dynamics of default and loss in the portfolio are modeled by an affine point process J with intensity $\lambda_t = \Lambda_0(t) + \Lambda_1(t) \cdot X_t$ driven by a risk factor X . Suppose further that J satisfies the growth conditions in Propositions 4.4 and A.1 and that the parameters of J have been calibrated to the multi-name market. Then if the thinning process Y^i is a deterministic function of time, the counting process N^i is an affine point process satisfying the conditions of Proposition 4.4 with intensity

$$\lambda_t^i = R_0(t) + R_1(t) \cdot X_t$$

⁵The default indicator process of firm i is $\bar{N}^i = N^i \wedge 1$. It has intensity $(1 - \bar{N}_-^i)\lambda^i$, extended intensity λ^i and thinning $(1 - \bar{N}_-^i)Y^i$ in the sense of Definitions 2.2 and 3.4 in Giesecke and Goldberg (2005).

where $R_0(t) = Y^i(t)\Lambda_0(t)$ and $R_1(t) = Y^i(t)\Lambda_1(t)$. Consequently, the survival probability (35) is exponentially affine in the state:

$$p_t^i(s) = e^{\alpha(\mathbf{0},t,s) + \beta(\mathbf{0},t,s) \cdot X_t} \quad (36)$$

where $\alpha(t) = \alpha(u, t, s)$ and $\beta(t) = \beta(u, t, s)$ satisfy the ordinary differential equations (31) and (32) are therefore functions of $Y^i(t)$. Thus, the parameters of $Y^i(t)$ are linked to the model survival probability (35) and they can be inferred from market implied survival probabilities.⁶

- (1) Specify a parametric model for each Y^i and simultaneously calibrate to market survival probabilities, noting summation and non-negativity constraints.
- (2) Shift the i th default swap spread by ΔS_t^i and compute the change in mark-to-market for the i th default swap:

$$\Delta M_t^i = \Delta S_t^i I^i V_t^i$$

where I^i is the notional of name i and V_t^i is the time t value of a dollar paid at premium times until the earlier of default or maturity; see formula (43).

- (3) Recalibrate the thinning process to the survival probabilities implied by ΔS_t^i . Here, we do not enforce the nonnegativity and summation constraints on the thinning. The change ΔY^i in the thinning generates a new affine point process satisfying the conditions of Propositions 4.4 and A.1. Its intensity is given by

$$(\Delta Y^i + 1)\lambda = (\Delta Y^i + 1)\Lambda_0 + (\Delta Y^i + 1)\Lambda_1 \cdot X. \quad (37)$$

- (4) Reprice the multi-name position with the shifted portfolio intensity (37). Proposition 4.4 can be applied directly. In the case of an index swap, the change in mark-to-market is given by

$$\Delta M_t = \Delta S_t I V_t$$

where ΔS_t is the change in the index spread due the shift in the i th swap spread, I is the index notional and V_t is the sum of the individual V_t^i s.

- (5) Estimate the delta hedge of the multi-name position with respect to name i as

$$\Delta_t^i \approx \frac{\Delta M_t}{\Delta M_t^i}.$$

⁶An additional assumption about the joint distribution of recovery and default is required to extract market survival probabilities from spreads. An alternative approach is to operate directly on the loss process, which already incorporates a model for this joint distribution.

A Proofs and additional results

Proof of Proposition 3.1. Define the function

$$f(t, x, \lambda) = E[e^{u \cdot J_s} | \mathcal{G}_t] = E[e^{u \cdot J_s} | J_t = x, \lambda_t = \lambda]$$

where the second equality is valid since X is Markov. From Revuz and Yor (2005, Chapter VII, Proposition 1.2),

$$\partial_t f + Af = 0. \quad (38)$$

where A is the infinitesimal generator of J . Formula (21) for the action of the transition semi-group of an affine point process on a bounded C^2 function g with bounded first and second derivatives reduces to

$$Ag(t, x, \lambda) = \lambda \int [g(t, x + \bar{z}, \lambda + \delta z) - g(t, x, \lambda)] d\nu(z) + \kappa(\lambda_\infty - \lambda) g_\lambda. \quad (39)$$

It follows that

$$\begin{cases} f_t + \lambda \int [f(t, x + \bar{z}, \lambda + \delta z) - f(t, x, \lambda)] d\nu(z) + \kappa(\lambda_\infty - \lambda) f_\lambda = 0 \\ f(s, x, \lambda) = e^{u \cdot J_s} \end{cases} \quad (40)$$

where $\bar{z} = (z, 1)^\top$ and subscripts on f denote partial derivatives. We make the following change of variables:

$$w(t, u, \lambda) = f(t, x, \lambda) e^{-u \cdot x}$$

Then (40) becomes

$$\begin{cases} w_t + \lambda \int [w(t, u, \lambda + \delta z) e^{u \cdot \bar{z}} - w(t, u, \lambda)] d\nu(z) + \kappa(\lambda_\infty - \lambda) w_\lambda = 0 \\ w(s) = 1 \end{cases}$$

Rearranging terms, we obtain

$$\begin{cases} w_t + \lambda \int [w(t, u, \lambda + \delta z) - w(t, u, \lambda)] e^{u \cdot \bar{z}} d\nu(z) \\ + \lambda \int (e^{u \cdot \bar{z}} - 1) w(t, u, \lambda) d\nu(z) + \kappa(\lambda_\infty - \lambda) w_\lambda = 0 \\ w(s) = 1 \end{cases}$$

This PIDE has an affine form in λ . We then propose the following form for w :

$$w(t, u, \lambda) = e^{a(t) + b(t)\lambda}$$

Then,

$$\lambda \partial_t b + \partial_t a + \lambda \int (e^{\delta b z} - 1) e^{u \cdot \bar{z}} d\nu(z) + \lambda \int (e^{u \cdot \bar{z}} - 1) d\nu(z) + \kappa(\lambda_\infty - \lambda) b = 0$$

$$\begin{cases} \partial_t b = \kappa b - \int (e^{\delta b z + u \cdot \bar{z}} - 1) d\nu(z) \\ \partial_t a = -\kappa \lambda_\infty b \end{cases}$$

with boundary conditions $a(s) = b(s) = 0$. □

Proof of Proposition 4.4. The result follows directly from Duffie et al. (2000, Proposition 1) since $(X, J)^\top$ is an affine jump diffusion. For readability, we adapt their argument to our framework. It suffices to show that Ψ is \mathbb{G} -martingale, for if so,

$$\begin{aligned} E[e^{-\int_t^s \rho(X_v, v) dv} e^{u \cdot J_s} | \mathcal{G}_t] &= e^{-\int_0^t \rho(X_v, v) dv} E[\Psi(s) | \mathcal{G}_t] \\ &= e^{-\int_0^t \rho(X_v, v) dv} \Psi(t) \\ &= e^{\alpha(u, t, s) + \beta(u, t, s) \cdot X_t + u \cdot J_t}, \end{aligned}$$

which is our assertion.

Applying Ito's formula and the Riccati equations (31) and (32), we see that Ψ is a local martingale since it is the sum of a driftless diffusion and a compensated jump process

$$\Psi(t) = \Psi(0) + \int_0^t \eta(v) dW_v + \int_0^t \Psi_v dN_v^\Psi - \int_0^t \gamma(v) dv.$$

where N^Ψ counts the jumps in Ψ . Condition (2) guarantees that the diffusion term is a martingale. As in Duffie et al. (2000, Appendix A), condition (3) guarantees that the compensated jump term is a martingale. \square

Proposition A.1. *For $y \in \mathbb{C}^d$, the conditional transform*

$$E[e^{-\int_t^s \rho(X_v, v) dv} y \cdot J_s | \mathcal{G}_t] = \psi(\mathbf{0}, X_t, J_t, t, s)(A(\mathbf{0}, t, s) + B(\mathbf{0}, t, s) \cdot X_t + y \cdot J_t)$$

if the coefficient functions $A(t) = A(u, t, s)$ and $B(t) = B(u, t, s)$ uniquely solve the ODEs

$$\begin{aligned} \partial_t B(t) &= -K_1^\top B(t) - \beta(t)^\top H_1 B(t) - \sum_{i=1}^m M_1^i \nabla \theta^i(\beta(t)) B(t) - \Lambda_1(\zeta B(t) + y) \cdot \nabla \theta(\zeta \beta(t)) \\ \partial_t A(t) &= -K_0 \cdot B(t) - \beta(t)^\top H_0 B(t) - \sum_{i=1}^m M_0^i \nabla \theta^i(\beta(t)) B(t) - \Lambda_0(\zeta B(t) + y) \cdot \nabla \theta(\zeta \beta(t)) \end{aligned}$$

with boundary conditions $A(s) = 0$ and $B(s) = \mathbf{0}$ and if for $(y, s) \in \mathbb{C}^d \times [0, \infty)$ the following expectations are all finite:

$$(1) E[|\Phi(y, X_s, J_s, 0, s)|]$$

$$\Phi(y, X_t, J_t, t, s) = \Psi(\mathbf{0}, X_t, J_t, t, s)(A(t) + B(t) \cdot X_t + y \cdot J_t)$$

$$(2) E\left[\left(\int_0^s |\tilde{\eta}(t)|^2 dt\right)^{1/2}\right]$$

$$\tilde{\eta}(t) = \Phi(y, X_t, J_t, t, s)(\beta(t)^\top + B(t)^\top) \sigma(X_t)$$

$$(3) E\left[\int_0^s |\tilde{\gamma}(t)| dt\right]$$

$$\begin{aligned} \tilde{\gamma}(t) &= \Phi(y, X_t, J_t, t, s) \left(\sum_{i=1}^m (\theta^i(\beta(t)) - 1) h^i(X_t) + (\theta(\zeta \beta(t) + y) - 1) \lambda(X_t, t) \right) + \\ &\Psi(\mathbf{0}, X_t, J_t, t, s) \left(\sum_{i=1}^m \nabla \theta^i(\beta(t)) \cdot B(t) + \nabla \theta(\zeta \beta(t) + y) \cdot \zeta B(t) \right) \end{aligned}$$

B The index swap spread from the bottom up

In Section 2.1 we derived the index spread (4) from the top down. The spread can alternatively be calculated from the bottom up as a weighted average of the premia S_t^i on single name swaps with common notional $I^i = I/n$. Since the index is simply a portfolio of single name default swaps with common maturity and premium payment dates, a no arbitrage argument implies that the value of the index premium leg must equal the sum over index constituents of the values of the single name premium legs.⁷ Thus, $S = S_t$ is the solution to the equation $P_t(S) = P_t(S_t^1, S_t^2, \dots, S_t^n)$ where

$$\begin{aligned} P_t(S^1, S^2, \dots, S^n) &= E \left[\sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m \sum_{i=1}^n S^i I^i (1 - N_{t_m}^i) \mid \mathcal{G}_t \right] \\ &= \frac{I}{n} \sum_{i=1}^n S^i \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m Q[\tau^i > t_m \mid \mathcal{G}_t] \end{aligned} \quad (41)$$

Formula (3) implies an analogous expression for the premium leg in terms of S_t :

$$P_t(S_t) = S_t \frac{I}{n} \sum_{i=1}^n \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m Q[\tau^i > t_m \mid \mathcal{G}_t]. \quad (42)$$

Let

$$V_t^i = \sum_{t_m \geq t} e^{-r(t_m-t)} \bar{\alpha}_m Q[\tau^i > t_m \mid \mathcal{G}_t] \quad (43)$$

be the value at time $t \leq T$ of a dollar paid at the premium times t_m unless firm i is in default. Equating equations (41) and (42) gives an expression for S_t in terms of the single name spreads and the V_t^i s:

$$S_t = \frac{\sum_{i=1}^n S_t^i V_t^i}{\sum_{i=1}^n V_t^i}. \quad (44)$$

The index spread at time t is a weighted average of the spreads of the single name default swaps in the index portfolio, with weights given by the corresponding V_t^i s.

C Random thinning and spread return

The change in the thinning $\Delta Y^i(t)$ has an intuitive meaning in the context of index spreads. Consider an investment grade index with a remaining term to maturity ($T - t$) of 1 year or less. Let the index notional be $I = 1$ and assume that the index portfolio

⁷This argument requires that the index swap and the single name swaps share a common set of credit events. In practice, the payoff for an index default swap is triggered only by bankruptcy or failure to make a payment. The payoff for a single name default swap may be triggered by a larger set of events.

intensity λ_t and the constituent intensities λ_t^i are constant over the remaining term. Letting $\ell = \int z d\nu(z)$ denote the expected value of the loss at default and assuming that risk-free interest rates are low, from formula (1) the value of the default leg of the index swap is given by

$$D_t \approx E[L_T - L_t | \mathcal{G}_t] = \ell E \left[\int_t^T \lambda_s ds \mid \mathcal{G}_t \right] = \ell \lambda_t (T - t).$$

If we approximate the discrete payments in the premium leg of the index swap with a continuous payment stream, then formula (42) becomes

$$P_t(S_t) = S_t \frac{1}{n} \sum_{i=1}^n \int_t^T e^{-r(s-t)} Q[\tau^i > s | \mathcal{G}_t] ds \approx S_t (T - t).$$

Since the index spread S_t equates the values of the default and premium legs, we get $\lambda_t \approx S_t/\ell$. From Step (3) in Section 5.2, the change in the portfolio intensity $\Delta\lambda_t$ due to a shift ΔS_t^i in the spread of name i is given by $\lambda_t Y^i(t)$. It follows that

$$\Delta Y^i(t) = \frac{\Delta\lambda_t}{\lambda_t} \approx \frac{\Delta S_t}{S_t}$$

so $\Delta Y^i(t)$ is a *spread return*; it is the relative change of the index spread due to a small change of the name i default swap spread.

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