THE MARKET PRICE OF CREDIT RISK

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September 23, 2003; this draft March 10, 2005‡

Abstract

We describe the relationship between physical probability of default and prices of credit sensitive securities. Our starting point is I^2, a first passage time model of default based on incomplete information. The I^2 model incorporates the unpredictable nature of default and thereby accounts for positive short spreads and the abrupt drops in prices of credit sensitive securities that occur at default. To connect prices with physical default probabilities, we analyze *post-default recovery* and the *credit risk premium.* Our recovery model is a generalization of the fractional market value convention introduced by Duffie & Singleton (1999) for intensity-based credit models. We derive generalized reduced-form pricing formulae for credit sensitive securities subject to fractional recovery. The credit risk premium has two components. One accounts for investors' aversion towards diffusive price volatility. The other reflects aversion toward the price jumps that occur at default, or more generally toward the default event itself. We conclude with a forward looking discussion of I^2 model calibration, and outline a strategy to extract the credit premium from market prices of credit sensitive securities.

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‡We thank Greg Anderson, Tim Backshall, Roveen Bhansali, Ursula Gritsch, Monique Jeanblanc, Alec Kercheval, Jyh-Huei Lee, Vijay Poduri, Philip Protter, Sergiy Terentyev and Yashan Wang for enlightening discussions. We are deeply grateful to Steve Evans for introducing us to Jacod's theory of martingale representation and explaining its relevance to the proof of Proposition 5.2. We are grateful to an anonymous referee, Robert Jarrow and Stefan Weber for thorough and insightful reviews of an earlier version of this article.

1 Introduction

Aggregate credit risk is one of the most pervasive threats in today's financial markets. It comes from the dependence across issuers of credit sensitive securities on the economic environment. It also arises from issuers interacting directly with their business partners, which can lead to a "contagious" propagation of distress. Taken together, these risks cannot be completely diversified away. It is a standard economic principle that undiversifiable risks commands a premium. In other words, risk-averse investors must be compensated for assuming undiversifiable credit risk.

The credit risk premium is empirically well-documented, see for example Elton, Gruber, Agrawal & Mann (2001), Collin-Dufresne, Goldstein & Helwege (2002), Collin-Dufresne, Goldstein & Martin (2001), Liu, Longstaff & Mandell (2000), Duffee (1999), Driessen (2002), or Pedrosa & Roll (1998). It is of central importance to financial practitioners and academicians since it connects the two main purposes of a credit model.

First, a credit model is used to forecast the probability of default. As such, the model must reflect the historical default experience. However, a credit model is also a tool for estimating the value of credit sensitive securities. In this context, it must fit observed market prices. In order to build a coherent model that serves both purposes, we need to understand the relationship between actual defaults and prices of credit sensitive securities. This is exactly where the risk premium comes into play. It maps the actual likelihood of default to the *pricing* or *martingale* likelihood of default that is used to price securities.

In this article, we analyze the credit risk premium in the context of I^2, which is a first passage time structural credit model described in Giesecke & Goldberg (2004*b*). First passage models have been considered in earlier works by Black & Cox (1976), Leland (1994), Longstaff & Schwartz (1995) and many others. In these models, credit risk is driven exclusively by uncertainty about the firm value and the risk premium takes a familiar form. The required excess return on any credit sensitive security issued by or referenced on the firm is equal to its risk times the market price of that risk. Here "risk" is measured in terms of diffusive price volatility. The market price of risk is given by the excess return on the firm per unit of firm risk.

However, this representation of the credit premium neglects the short-term uncertainty surrounding the default event. Indeed, in the models cited above, the distance of the firm to default is always observable, so default is predictable. The existence of short-term uncertainty in the credit market is highlighted by the prevalence of positive short-term credit spreads and the precipitous drops in equity and bond prices that occur at default. Empirical observation shows that equity drops to near zero. Bondholders usually lose something but generally do not lose everything. Consequently, net firm value, which is equal to the sum of equity and debt values, also drops at default. In order to fit market prices, a credit model must take account of these discontinuities.

It is shown in Duffie & Lando (2001), Giesecke (2001), Çetin, Jarrow, Protter & Yildirim (2004) and Giesecke & Goldberg (2004*b*) that the forecasts of a structural models become more realistic if the assumption that investors are completely informed is relaxed. In these incomplete information models, the distance to default cannot be observed by investors and default is a totally inaccessible event. As reduced-form credit models, incomplete information models admit short-term credit risk. The jumps in security prices induced by this short-term uncertainty may command a premium.

To analyze security prices and the credit premium in I^2, we extend its scope to include the downward jump in security prices observed at default. To do this, we introduce a stochastic model for post default recovery that is described, along with our other model assumptions, in Section 2. Our approach is based on the *fractional recovery* convention introduced by Duffie & Singleton (1999) for intensity-based reduced-form models. In Section 7, we extend their approach to a structural, incomplete information setting and give generalized reduced-form security pricing formulae in the spirit of Giesecke (2001). These convenient pricing results simplify to the formulae in Duffie & Singleton (1999) when the model admits an intensity. The I^2 model does not have an intensity.

The risk premium corresponds to a pricing measure, which can be represented by its density with respect to the physical measure. In Section 5, we show that the space of densities for I^2 is parameterized by pairs of processes that are predictable in the investors' filtration. Thus, as discussed in Section 6, the credit risk premium on a credit sensitive security can always be decomposed into two economically meaningful components. The *diffusive risk premium*, which is realized as a change to the drift term in the security's price process, is proportional to the security's diffusive price volatility. The proportionality factor can be interpreted as a market price of diffusion risk in the firm value. There is also a *default event risk premium*, which accounts for investors' risk aversion towards the downward jump in prices upon default. It prescribes the mapping between default probabilities under the physical measure and the pricing measure. Driessen (2002), Collin-Dufresne et al. (2002) and Berndt, Douglas, Duffie, Ferguson & Schranz (2004) empirically confirm that this event risk premium is a significant factor in corporate bond returns.

This economic picture is based on the explicit parametrization of the full space of martingales on our probability space. To characterize this space, El Karoui & Martellini (2001) and Bielecki & Rutkowski (2002) sketch monotone class arguments that we were unable to complete. Our analysis uses the powerful martingale representation results of Jacod (1977). We show that every uniformly integrable martingale can be represented in terms of a Brownian motion that drives the diffusion-type uncertainty in the net firm value and the compensated default jump martingale, which represents the jump-type uncertainty in the net firm value. The risk premia are proportional to the martingale coefficients in this representation. This extends the similar representation result of Kusuoka (1999).

The I^2 model is a hybrid that has attributes in common with traditional structural models and reduced form models. This hybrid nature extends to the structure of the space of risk premia. The diffusive risk premium in I^2 is analogous to the risk premium for traditional structural models. The default event risk premium in I^2 is not present in traditional structural models. However, it is analogous to the default event risk premium in intensity-based models considered in Kusuoka (1999), El Karoui & Martellini (2001), Jarrow, Lando & Yu (2005), Driessen (2002) and Collin-Dufresne et al. (2002).

The paper is organized as follows. In Section 2, we outline our model assumptions and some immediate consequences. The underlying probabilistic structure is discussed in the Appendix. In Section 3, we summarize the discussion in Giesecke & Goldberg (2004*b*) of I^2 default probability forecasts. In Section 4, we review the economic principles that enable us to extend our default forecasting model to become a security pricing tool. Sections 3 and 4 contain no new material; they are included for readability and completeness. Section 5 gives a complete, mathematical analysis of the space of equivalent martingale measures. This material is recast in terms of the risk premium in Section 6. We explain how to price credit sensitive securities in Section 7. In the concluding Section 8, we outline a model calibration strategy based on the ideas in this article. This strategy is implemented in Giesecke & Goldberg (2004*a*), where we calibrate I^2 simultaneously to physical and martingale measures using pooled data from equity and credit markets.

2 The I^2 Model

2.1 Assumptions

We model the uncertainty in the economy with a filtered complete probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0\leq t\leq \bar{T}}, \pi)$, where $\bar{T} > 0$ is some large but finite horizon. The elements of the set Ω represent the possible states of nature and $\mathcal{G}$ is a sigma-algebra of subsets of Ω. By identifying an occurrence with the set of states in which it occurs, the elements of $\mathcal{G}$ correspond to events. The filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq \bar{T}}$ is a non-decreasing family of sigma-algebras contained in $\mathcal{G}$; it models the evolution of information in the economy. The symbol π denotes a probability measure that gives the likelihood of any event in $\mathcal{G}$.

We consider a fixed firm in the economy and make the following assumptions. The probabilistic structure underlying these assumptions is discussed in more detail in Appendix A.

A1. Capital structure of the firm:
The firm is financed by equity and debt. Debt is senior to equity.

A2. Gross firm value:

The gross firm value X_t is the present value at time t of all future cash flows generated by the firm. It follows a geometric Brownian motion under the measure π. This is described by the equation

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t, \quad X_0 > 0, \tag{1}$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion with respect to the measure π and its augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(W_s : s \leq t)$ where $\mathcal{F}_0$ is the collection of null sets in $\mathcal{G}$. The equation (1) has the unique strong solution

$$X_t = X_0 e^{V_t} \tag{2}$$

where $V_t = mt + \sigma W_t$ is a Brownian motion with drift $m = \mu - \frac{1}{2}\sigma^2$.

A3. Default time:

The firm's management decides whether to default. We assume that the firm defaults if the gross firm value X falls to some barrier. This *default threshold* is modeled by a random variable $d \in (0, X_0)$, which we assume is independent of $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. In what follows, it is convenient to use the normalized default threshold $D = \log(d/X_0)$ that lies in the interval $(-\infty, 0]$. We can then write the random default time τ as

$$\tau = \inf\{t > 0 : V_t \leq D\}. \tag{3}$$

Associated to the default time τ is the *indicator process* N defined as

$$N_t = 1_{\{\tau \leq t\}}. \tag{4}$$

A4. Information structure:

Investors in public bonds observe the gross firm value and defaults but not the level d. Therefore, their information is *incomplete*. The value of d depends on the firm's liabilities and is assumed to be firm inside information. In mathematical terms, the public information flow is modeled by the augmented, right continuous[1] filtration $\mathbb{G}$ generated by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(N_s : s \leq t). \tag{5}$$

A5. Threshold prior:

Lacking definite knowledge of the default point, investors agree on a prior distribution function G on the normalized default threshold D. We assume G has density g.

[1] See Bélanger, Shreve & Wong (2004, Appendix A) for a proof of right continuity.

A6. Credit sensitive claims:

The firm has issued credit sensitive claims including equity and debt. A general claim is characterized by its payoff $c_T \in L^1(\Omega, \mathcal{F}_T, \pi)$ at a horizon $T \leq \bar{T}$. The payoff is made if there was no default by T.[2] Mathematically, the time T payoff is

$$C_T = c_T 1_{\{\tau > T\}}. \tag{6}$$

If the firm defaults, a recovery payment is made. We follow the fractional recovery convention introduced by Duffie & Singleton (1999) in the context of intensity-based models. Let R be an $\mathbb{F}$-predictable process with values in $[0, 1]$. If the firm defaults at time t, a fraction R_t of the market value of the claim just prior to default is recovered. The fraction $1 - R_t$ represents *bankruptcy costs*. Therefore, the payoff at the default time is

$$C_\tau = R_\tau \cdot C_{\tau-} \cdot 1_{\{\tau \leq T\}}. \tag{7}$$

A credit sensitive claim is characterized by the triple (T, c_T, R).

If (T, c_T, R) denotes the bond issue, then T is its maturity date, c_T is its face value, and R is its non-trivial recovery process. In this case we denote the bond price process by B. If (T, c_T, R) denotes the corresponding equity claim, then $R = 0$ and c_T is equity investors' payoff at debt maturity T. We denote the equity price process by S.

A7. Dynamics of credit sensitive claim prices:

The value dynamics of the credit sensitive claim (T, c_T, R) with respect to the filtration $\mathbb{G}$ are described by the stochastic differential equation

$$\frac{dC_t}{C_{t-}} = d\mu_C(t) + \sigma_C(t)dW_t - (1 - R_t)dN_t, \quad C_0 > 0. \tag{8}$$

Here, $\mu_C = (\mu_C(t))_{t \geq 0}$ is a $\mathbb{G}$-adapted process that starts at zero and has continuous paths of finite variation. It describes the *cumulative* growth rate of C. Further, $\sigma_C = (\sigma_C(t))_{t \geq 0}$ is a strictly positive $\mathbb{G}$-predictable process that describes the diffusive volatility of C. The processes μ_C and σ_C are such that (8) is well-defined.

If the claim (T, c_T, R) denotes equity or bonds, then we denote the pair (μ_C, σ_C) by (μ_S, σ_S) or (μ_B, σ_B), respectively.

A8. Riskless bonds:

On the financial market investors can trade in riskless bonds. Given some constant riskless rate r, these are valued at e^{rt} at time t.

[2] The choice $c_T \in \mathcal{F}_T$ is without loss of generality: For any $\mathcal{G}_T$-measurable random variable c, we have $c1_{\{\tau > T\}} = \bar{c}1_{\{\tau > T\}}$, where $\bar{c}$ is $\mathcal{F}_T$-measurable.

2.2 Consequences

The model assumptions outlined above have several consequences which are important in the sequel. Once again, Appendix A provides more details with respect to the underlying probabilistic structure.

C1. Observability of defaults:
Assumption A4 implies that the random default time τ is a *stopping time* in the filtration $\mathbb{G}$. It means that at each point in time, the default status of the firm can be observed.

C2. Unpredictability of defaults:
Assumptions A3 and A4 imply that the default time τ is *totally inaccessible* in the filtration $\mathbb{G}$. In mathematical terms, $\pi[\tau = \sigma < \infty] = 0$ for all $\mathbb{G}$-predictable times σ. On an intuitive level, it means that default cannot be anticipated. This is economically reasonable. Since investors are not privileged to firm inside information, they do not know the true distance between gross firm value and the default barrier. The unpredictability of defaults is consistent with the sudden downward jumps in the market value of debt and equity.

C3. Gross firm value and survival information:
The $\mathbb{F}$-Brownian motion W is also a Brownian motion in investors' filtration $\mathbb{G}$. This is a consequence of the independence of W and the default barrier, see A3. A formal discussion is in Appendix A.

It follows that any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale. In other words, a process that is fair with respect to the information described by the augmented Brownian filtration $\mathbb{F}$ remains fair in the larger filtration $\mathbb{G}$, which contains survival information in addition.

C4. Net firm value:
The net firm value $\mathcal{X}$ is the sum of the values of equity S and debt B. Assumption A4 implies that the dynamics of $\mathcal{X}$ in the filtration $\mathbb{G}$ are of the form (8). They are described by the equation

$$\frac{d\mathcal{X}_t}{\mathcal{X}_{t-}} = d\mu_{\mathcal{X}}(t) + \sigma_{\mathcal{X}}(t)dW_t - J_t dN_t, \quad \mathcal{X}_0 > 0. \tag{9}$$

If we let B denote the value of all debt securities issued by the firm, then the cumulative growth rate $\mu_{\mathcal{X}}$ of the net firm value satisfies

$$d\mu_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-}d\mu_S(t) + B_{t-}d\mu_B(t)\big)$$

and the volatility of the net firm value $\sigma_{\mathcal{X}}$ can be expressed as

$$\sigma_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-}\sigma_S(t) + B_{t-}\sigma_B(t)\big).$$

At default, $\mathcal{X}$ jumps downwards, mirroring the losses of equity and bonds. If default were to occur at time t, the combined losses relative to $\mathcal{X}$ are

$$J_t = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-} + (1 - R_t) \cdot B_{t-}\big).$$

The value $J_t \mathcal{X}_{t-}$ represents the costs of bankruptcy, see A6. This value is lost to third parties at default. Thus, the net firm value $\mathcal{X}_t$ differs from the gross firm value X_t, which represents the present value of the future cash flows generated by the firm (A2). It follows that the I^2 model assumptions A1-A8 are not consistent with the Modigliani-Miller theorem. See Giesecke & Goldberg (2004*c*) for a discussion.

3 Probability of Default

We are interested in estimates of the conditional probability of default with respect to the investor filtration $\mathbb{G}$ under the assumptions A1-A8. Since these assumptions imply that the default time is totally inaccessible (C2), we can apply the generalized reduced-form theory developed in Elliott, Jeanblanc & Yor (2000), Giesecke (2001), Bélanger et al. (2004) and others.

3.1 Reduced-form representation

Consider the *conditional survival probability* in the filtration $\mathbb{F}$, defined by

$$L_t = E^\pi[1 - N_t \,|\, \mathcal{F}_t] = \pi[\tau > t \,|\, \mathcal{F}_t]. \tag{10}$$

It describes the default process N in terms of the information contained in the filtration $\mathbb{F}$. Define the process F by

$$F_t = -\log L_t.$$

The $\mathbb{F}$-adapted process F leads to reduced form formulae for $\mathbb{G}$-conditional default probabilities. For some given time t, fix some horizon $T \in [t, \bar{T}]$. Assume that $L_t > 0$ for all t. On the no-default set $\{t < \tau\}$, we have the formula

$$\pi[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E^\pi[e^{F_t - F_T} \,|\, \mathcal{F}_t], \tag{11}$$

see Elliott et al. (2000, Proposition 3.1). On $\{t < \tau\}$, the conditional default probability with respect to the investor filtration $\mathbb{G}$ is expressed in terms of an $\mathbb{F}$-conditional expectation.

3.2 Compensator and trend

The survival probability process L is an $\mathbb{F}$-supermartingale, and the Doob-Meyer decomposition of L asserts that there exists a unique non-decreasing $\mathbb{F}$-predictable process K, called the compensator of L, such that $L + K$ is an $\mathbb{F}$-martingale, cf. Dellacherie & Meyer (1982, VII, Theorem 8).

Definition 3.1. *The non-decreasing $\mathbb{F}$-predictable process A^π defined by*

$$A_t^\pi = \int_0^t \frac{dK_s}{L_{s-}} \tag{12}$$

is called the trend *of N under π.*

If L has monotone and continuous paths, then $K = 1 - L$ and we have $A^\pi = F$. If L has paths that are absolutely continuous with respect to the Lebesgue measure, then so are the paths of A^π and we have

$$A_t^\pi = \int_0^t \lambda_s^\pi ds \tag{13}$$

for some density process λ^π, which is called the $\mathbb{F}$-*intensity* of τ. It describes the $\mathbb{F}$-conditional default rate.

The default indicator process N is a $\mathbb{G}$-submartingale. It admits a unique Doob-Meyer decomposition into the sum of a $\mathbb{G}$-martingale and a non-decreasing $\mathbb{G}$-predictable process called the compensator of N. The following result is due to Jeulin & Yor (1978).

Proposition 3.2. *The process H defined by*

$$H_t = N_t - A_{t\wedge\tau}^\pi \tag{14}$$

is a $(\pi, \mathbb{G})$-martingale. In other words, $A_{\cdot\wedge\tau}^\pi$ is the $\mathbb{G}$-compensator of N.

3.3 I^2-trend and default probability

Consider the conditional survival probability under the assumptions A1-A8. Since the assets and default threshold are independent, (3) implies that

$$L_t = \pi[D < M_t \,|\, \mathcal{F}_t] = G(M_t), \tag{15}$$

where M is the $\mathbb{F}$-adapted process of the historical log-asset lows:

$$M_t = \min_{s\leq t} V_s. \tag{16}$$

Since L is monotone and continuous, the trend A^π is given by the formula

$$A_t^\pi = F_t = -\log G(M_t). \tag{17}$$

Since M is of finite variation and G has density g, we get

$$dA_t^\pi = -\frac{g(M_t)}{G(M_t)}dM_t.$$

This equation shows that A^π increases only when $-M$ does, which is when $M_t = V_t$ and assets reach their historical low. The set of times $\{t \geq 0 : M_t = V_t\}$ has Lebesgue measure zero, and therefore the trend does not admit an intensity in the sense of (13).

Example 3.3 (Giesecke (2001)). *Suppose that the default threshold d is* uniform *on $(0, X_0)$. Then the normalized barrier D has distribution function $G(x) = e^x$ on $(-\infty, 0)$ and*

$$\pi[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E^\pi[e^{M_T - M_t} \,|\, \mathcal{F}_t] = p(T-t, V_t - M_t),$$

where for $s > 0$ and $v \geq 0$ we define

$$p(s,v) = \int_{-\infty}^{-v} \Psi(s,y)e^{y+v}dy$$

with

$$\Psi(s,v) = \Phi\left(\frac{v - mt}{\sigma\sqrt{s}}\right) + \exp\left(\frac{2mv}{\sigma^2}\right)\Phi\left(\frac{v+mt}{\sigma\sqrt{s}}\right).$$

Here, Φ denotes the standard normal distribution function. Defining furthermore the constants $\nu = m + \sigma^2$, $\mu = m + \sigma^2/2$, $\gamma = 1 + 2m/\sigma^2$, $\delta = m - \gamma\sigma^2$, and $\eta = -m\gamma + \gamma^2\sigma^2/2$, by integration by parts we find

$$\begin{aligned} p(s,v) = {} & \Phi\left(\frac{-v - ms}{\sigma\sqrt{s}}\right) - e^{v+s\mu}\Phi\left(\frac{-v-\nu s}{\sigma\sqrt{s}}\right) \\ & + \frac{1}{\gamma}e^{(1-\gamma)v}\Phi\left(\frac{ms - v}{\sigma\sqrt{s}}\right) - \frac{1}{\gamma}e^{v+s\eta}\Phi\left(\frac{\delta s - v}{\sigma\sqrt{s}}\right). \end{aligned} \tag{18}$$

Example 3.4 (Giesecke & Goldberg (2004*b*)). *Suppose $\kappa \in (0, X_0)$ and the scaled default threshold d/κ follows a* Beta distribution *with parameters $a > 1$ and $b > 1$. Then the normalized default barrier D has distribution function*

$$G(x) = \frac{c(a,b)X_0}{\kappa}\int_{-\infty}^{x}\left(\frac{X_0e^u}{\kappa}\right)^{a-1}\left(1 - \frac{X_0e^u}{\kappa}\right)^{b-1}e^u du$$

on $(-\infty, 0)$ and the probability of default is given by

$$\begin{aligned} & \pi[\tau \leq T \,|\, \mathcal{G}_t] \\ & = 1 - \frac{1}{G(M_t)}\int_{-\infty}^{M_t} G(y)\frac{d}{dv}\Psi(T-t, y - V_t)dy \\ & = 1 - \frac{c(a,b)X_0}{\kappa G(M_t)}\int_{-\infty}^{M_t \wedge \log(\frac{\kappa}{X_0})}\left(\frac{X_0e^u}{\kappa}\right)^{a-1}\left(1 - \frac{X_0e^u}{\kappa}\right)^{b-1}e^u\Psi(T-t, u - V_t)du. \end{aligned}$$

4 Pricing principles

We investigate the relationship between the probability of default and the price of a credit sensitive claim. Consider a claim $(T, c_T, 0)$ for some horizon $T \leq \bar{T}$ and integrable random variable $c_T \in \mathcal{F}_T$. What is a fair price C_t for such a claim? From an actuarial point of view, C_t should be equal to the expected value of the payoff discounted to time t. According to the *actuarial principle*, the time t value of the claim is

$$C_t = e^{-r(T-t)} E^{\pi}[c_T \cdot 1_{\{\tau > T\}} \,|\, \mathcal{G}_t] \tag{19}$$

where π is taken to be the *physical* measure, denoted by P. So under the actuarial principle the price of a defaultable zero coupon bond where $c_T = 1$, is just given by $C_t = e^{-r(T-t)} P[\tau > T \,|\, \mathcal{G}_t]$. All we need is the actual P-survival probability, which we computed in the previous section.

Although this principle seems natural and convenient, it has significant deficiencies. For instance, it implies that the difference in price between a credit sensitive claim and an otherwise identical claim carrying no credit risk covers only the expected loss due to default. In fact, most investors are risk-averse and demand extra compensation, or a *risk premium*, for assuming the risk of losses.

To account for risk aversion, the actuarial scheme must be modified to generate higher compensation to the investor or equivalently, lower security prices. The standard approach is to retain the form of the principle (19) and to substitute a *pricing* measure for the reference measure π. Events such as "no default by time T" are assigned new probabilities that do not necessarily reflect the actual likelihood of survival. Rather, they are consistent with market prices. When π is a pricing measure, formula (19) generates a price that accounts for both the expected default loss and the risk premium.

A pricing measure is characterized by two properties.

M1. Martingale property:
The discounted price process $(C_t e^{-rt})$ of any traded credit sensitive security must be a $\mathbb{G}$-martingale with respect to the pricing measure.

M2. Equivalence:
The pricing measure and physical measure belong to the same class. In other words, they agree on which sets in $\mathcal{G}_T$ have zero measure.

Let $\mathcal{P}$ denote the set of measures on $(\Omega, \mathcal{G}_T)$ satisfying M1 and M2. The measures in $\mathcal{P}$ are called *pricing measures* or *equivalent martingale measures*. We use the terms interchangeably.

The mathematical conditions determining $\mathcal{P}$ arise from a fundamental economic result in Delbaen & Schachermayer (1997) that goes back to Harrison & Kreps (1979) and Harrison & Pliska (1981): Under broad assumptions, $\mathcal{P}$ is non-empty if and only if the security prices generated by the elements in $\mathcal{P}$ do not admit arbitrage opportunities. Further, $\mathcal{P}$ consists of a single measure if and only if markets are complete. These deep results point to the most serious deficiency of the actuarial pricing principle: it does not guarantee the absence of arbitrage opportunities. In fact, if markets are complete and the risk premium is non-trivial, the actuarial principle implies an arbitrage.

All pricing measures account for default risk and each one accounts for the risk premium in its own way. If the financial market is arbitrage-free but incomplete, there are infinitely many martingale measures and thus, infinitely many risk premia. This is because $\mathcal{P}$ is convex and has more than one element.

In the incomplete case, conditions M1 and M2 do not lead to a unique price but to an arbitrage-free price interval $\big(\inf_{\pi\in\mathcal{P}} C_t(\pi), \sup_{\pi\in\mathcal{P}} C_t(\pi)\big)$. This interval is usually too wide to be of practical use for the pricing of credit sensitive claims. In this situation, we may impose additional criteria in order to select a specific element from $\mathcal{P}$ for pricing. Two useful criteria are the minimization of residual hedging risk (Föllmer & Sondermann (1986)) and the minimization of the relative entropy between the martingale measure and the physical measure (Föllmer & Schweizer (1990)). Alternatively, we can obtain a pricing measure from an equilibrium model of the financial market. We refer to Chang & Sundaresan (1999) for the first steps in this direction. From a practical viewpoint it is often convenient to directly extract the equilibrium pricing measure from observed market prices of traded credit sensitive claims. Any of these strategies requires a detailed understanding of the set of martingale measures, which is the subject of the next section.

5 Pricing measures

5.1 Overview

Throughout this section, the reference measure π is the physical measure P. We examine the set $\mathcal{P}$ of martingale measures equivalent to P in more detail. Our analysis sheds light on the structure of the risk adjustment, which provides an economic link between P and the measures in $\mathcal{P}$.

The space $\mathcal{P}$ sits inside the set $\mathcal{E}$ of measures that are equivalent to the physical measure.[3] In other words, the measures in $\mathcal{E}$ satisfy M2 but may not satisfy M1. We start with the observation that each $Q \in \mathcal{E}$ can be identified with a P-martingale $Z = Z(Q)$. In Theorem 5.1, we show that $Z(Q)$ can be represented as a particular function of a pair of $\mathbb{G}$-predictable processes α and β. Thus, the space $\mathcal{E}$ is parameterized by

[3]Note that the physical measure is in $\mathcal{E}$ but not usually in $\mathcal{P}$. In cases where the physical measure is in $\mathcal{P}$ the risk premium is zero.

the pairs (α, β). The representation in Theorem 5.1 depends on the filtration $\mathbb{G}$ and the measure class of P but not on which securities are traded.

The next step is to express the Q-price processes of traded securities in terms of its associated P-martingale Z. Direct analysis of the Theorem 5.1 representation for $Z = Z(\alpha, \beta)$ gives rise to necessary and sufficient conditions on α and β for which the price processes are martingales. These conditions are given in Theorem 5.3.

5.2 Equivalent measures

Fix a horizon $T \leq \bar{T}$. Let $L^1 = L^1(P)$ denote the P-integrable functions on $(\Omega, \mathcal{G}_T)$. The relationship between the physical measure P and any $Q \in \mathcal{E}$ is expressed in terms of a positive random variable $Z_T = dQ/dP \in L^1$ for which $E^P[Z_T] = 1$. The variable Z_T is called the Radon-Nikodym derivative of Q with respect to P. Let Z be the right-continuous version of the P-martingale defined by

$$Z_t = E^P[Z_T \,|\, \mathcal{G}_t], \quad t \leq T. \tag{20}$$

Then Z satisfies the equation

$$E^Q[Y|\mathcal{G}_t] = \frac{1}{Z_t} E^P[Y Z_T|\mathcal{G}_t] \tag{21}$$

for all bounded $Y \in \mathcal{G}_T$ and $t \leq T$.

Theorem 5.1. *For all $t \leq T$ the density Z_t is of the form*

$$Z_t = \exp\left(-\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2 ds + \int_0^t \log(1+\beta_s) dN_s - \int_0^{t\wedge\tau} \beta_s dA_s^P \right) \tag{22}$$

where α and β are $\mathbb{G}$-predictable and $\beta > -1$ almost surely. The processes α and β satisfy

$$E^P\left[\int_0^{T_n} \alpha_s^2 ds\right] < \infty \quad \text{and} \quad E^P\left[\int_0^{T_n} |\beta_s| dA_s^P\right] < \infty \tag{23}$$

for a sequence of $\mathbb{G}$-stopping times T_n that increase to T. There is an $\mathbb{F}$-predictable process $\bar{\beta}$ that agrees with β on $\{\tau > t\}$.

Conversely, $\mathbb{G}$-predictable processes α and β that satisfy (23) and for which the density $Z_T = Z_T(\alpha, \beta)$ defined by (22) satisfies $E^P[Z_T] = 1$, correspond to a measure $Q = Q(\alpha, \beta) \in \mathcal{P}$.

Equation (22) represents the martingale Z in terms of the two martingales that generate the uncertainty in our model. These are the Brownian motion W that underlies the gross firm value process and the compensated jump martingale H introduced in equation (14). The processes α and β in Theorem 5.1 are used to convert the measure P to an equivalent measure Q with respect to which the processes W and

H are no longer driftless. In Section 6, we characterize α and β as components of the risk premium.

Kusuoka (1999) proves a result similar to Theorem 5.1 under the additional assumption that the martingale Z is square integrable.[4] Our result for I^2 is not subject to this restriction.

The proof of Theorem 5.1 is based on the following proposition.

Proposition 5.2. *The martingale Z can be expressed as a sum*

$$Z_t = 1 + \int_0^t a_u dW_u + \int_0^t b_u dH_u, \tag{24}$$

where a and b are $\mathbb{G}$-predictable processes. These processes satisfy

$$E^P\left[\int_0^{T_n} a_s^2 ds\right] < \infty \quad \text{and} \quad E^P\left[\int_0^{T_n} |b_s| dA_s^P\right] < \infty \tag{25}$$

for a sequence of $\mathbb{G}$-stopping times T_n that increase to T almost surely.

Proof. For $Z_T \in L^2(\Omega, \mathcal{G}_T, P)$, the representation (24) holds under more stringent growth conditions on a and b:

$$E^P\left[\int_0^{T} a_s^2 ds\right] < \infty \quad \text{and} \quad E^P\left[\int_0^{T} b_s^2 dA_s^P\right] < \infty.$$

See for example Kusuoka (1999). Here, we follow a different line of reasoning to obtain a broader result under weaker restrictions on the coefficients.

Let M_P be the collection of all martingales with respect to the stochastic basis $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ that take the form (24) and satisfy (25).

A special case of Jacod (1977, Theorem 2) is that if P is the unique measure on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T})$ for which every element of M_P is a martingale, then every martingale on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ is in M_P. We consider the subset $\mathcal{M}_P$ of M_P that consists of $(W_t, H_t, W_t^2 - t)_{0\le t\le \bar{T}}$. It suffices to show that P is unique in the sense above with respect to the elements in $\mathcal{M}_P$.

Suppose that $P' \in M_P$. Since W and $(W_t^2 - t)$ are continuous martingales for both P and P' with respect to $\mathbb{G}$, Lévy's theorem implies that W is a Brownian motion under both measures. It follows that $P(W_{t_i} \in A_i; i = 1, 2, \dots, n) = P'(W_{t_i} \in A_i; i = 1, 2, \dots, n)$ for Borel sets A_i so that $P = P'$ on sets in $\mathcal{F}_T$.

Further, since H is a martingale for both P and P', the uniqueness of the Doob-Meyer decomposition implies that $A^P_{\cdot\wedge\tau}$ is the compensator of N for both P and P'. Let A^P and $A^{P'}$ be the $\mathbb{F}$-predictable trends of N under P and P' respectively. From (14) and (17),

$$A^P_{t\wedge\tau} = A^{P'}_{t\wedge\tau} \tag{26}$$

[4]Kusuoka (1999) also assumes that the trend A^P admits an intensity in the sense of (13). The proof of his representation Theorem 2.3 does however not require this assumption.

so that A^P and $A^{P'}$ agree for $t \leq \tau$. We show that they agree on $[0, \infty)$ almost surely. Let Γ be the infimum of all times at which the trends A^P and $A^{P'}$ disagree. Then Γ is an $\mathbb{F}$-stopping time which is an upper bound for τ. This means the running minimum log-firm value M_Γ is less than the default barrier D. But D is not observable in the filtration $\mathbb{F}$. It follows that $\Gamma = \infty$ almost surely and the trends A^P and $A^{P'}$ are indistinguishable.

Let $U \in \mathcal{F}_T$. Then

$$E^{P'}[1_U(1 - N_t)] = E^{P'}[1_U e^{-A_t^{P'}}] = E^{P'}[1_U e^{-A_t^P}] = E^P[1_U e^{-A_t^P}]$$

where the first equation follows from Giesecke (2001, Theorem 4.5) and the third equation follows from the fact that the argument of the expectation is $\mathcal{F}_T$-measurable. Since every set in $\mathcal{G}_T$ can be arbitrarily well approximated by finite unions and complements of sets of the form $U \cap \{\tau \leq t\}$, it follows that P and P' agree on $\mathcal{G}_T$. Thus, P is the unique measure for which the processes in $\mathcal{M}_P$ are martingales.

Now, Jacod (1977, Theorem 2) implies that every martingale on the space $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0 \leq t \leq T}, P)$ can be represented as in equation (24) with coefficients satisfying conditions (25). $\square$

We now give the proof of Theorem 5.1.

Proof. It is shown in Jacod (1979) that if Z is a positive process, the right continuous left limit process Z_- is also positive. Further, the process $1/Z_-$ is locally bounded. To see this, let Γ_n be the first time that $Z_- \leq 1/n$. If $1/Z_-$ is not locally bounded, then the $\mathbb{G}$-stopping times Γ_n increase to a stopping time Γ that is strictly less than T on a set $U \in \mathcal{G}_T$ of positive measure. But then $E^P[Z_{\Gamma-} 1_U] = 0$, contradicting the fact that $Z_- > 0$.

Let $\alpha = a/Z_-$ and $\beta = b/Z_-$ where a and b are the $\mathbb{G}$-predictable processes defined in Proposition 5.2. Since Z_- is locally bounded, α and β satisfy (23) and we can define a semimartingale Y by

$$Y_t = -\int_0^t \alpha_s dW_s + \int_0^t \beta_s dH_s, \quad t \leq T. \tag{27}$$

Equation (24) can be rewritten

$$Z_t = 1 + \int_0^t Z_{s-} dY_s, \quad t \leq T. \tag{28}$$

It is well-known (see, for example Protter (2004, Chapter II, Theorem 36)), that (28) admits the unique solution

$$Z_t = \exp\left(Y_t - \frac{1}{2}[Y, Y]_t^c\right) \prod_{s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s), \tag{29}$$

called the stochastic exponential of Y. Here, $[Y,Y]^c$ is the (path-by-path) continuous part of the quadratic variation process of Y, and $\Delta Y_t = Y_t - Y_{t-}$ is the jump of Y at time t.

Using Protter (2004, Chapter IV, Theorem 22), we find that

$$[Y,Y]_t = [-\int_0^{\cdot} \alpha_s dW_s + \int_0^{\cdot} \beta_s dH_s, -\int_0^{\cdot} \alpha_s dW_s + \int_0^{\cdot} \beta_s dH_s]_t \tag{30}$$

$$= \int_0^t \alpha_s^2 d[W,W]_s - 2\int_0^t \alpha_s\beta_s d[W,H]_s + \int_0^t \beta_s^2 d[H,H]_s \tag{31}$$

$$= \int_0^t \alpha_s^2 ds + \int_0^t \beta_s^2 dN_s, \tag{32}$$

since $[W,W]_t = t$ by Lévy's theorem, $[W,H] = 0$, and $[H,H] = [N - A^P_{\cdot\wedge\tau}, N - A^P_{\cdot\wedge\tau}] = [N,N] = N$ because $A^P_{\cdot\wedge\tau}$ is of finite variation. Now clearly

$$[Y,Y]^c_t = \int_0^t \alpha_s^2 ds.$$

Since the default stopping time τ is totally inaccessible, the compensator of the default indicator N is continuous, cf. Dellacherie & Meyer (1982). It follows that H is continuous except for a jump of size 1 at τ. Therefore, $\Delta Y_t = \beta_\tau 1_{\{t=\tau\}}$, i.e. ΔY is zero except at time τ, where it is equal to β_τ.

From (29) we now obtain

$$Z_t = \exp\left(Y_t - \frac{1}{2}\int_0^t \alpha_s^2 ds\right)(1 + \beta_\tau 1_{\{\tau\le t\}})\exp(-\beta_\tau 1_{\{\tau\le t\}}). \tag{33}$$

Noting that

$$\beta_\tau 1_{\{\tau\le t\}} = \int_0^t \beta_s dN_s,$$

after substituting for Y and rearranging we obtain our assertion.

For the $\mathbb{G}$-predictable process β there exists a $\mathbb{F}$-predictable process $\bar\beta$ such that $\beta_t = \bar\beta_t$ on the set $\{t \le \tau\}$, cf. Jeulin & Yor (1978). Since the process β is only of interest before default, we can, without loss of generality, choose it to be $\mathbb{F}$-predictable.

To prove the converse, suppose that there are $\mathbb{G}$-predictable processes α and β satisfying the conditions (23). Let $Z_T = Z_T(\alpha,\beta)$ be defined by (22). Then $Z_T > 0$ and if $E^P[Z_T] = 1$, it is the density dQ/dP of some equivalent measure $Q = Q(\alpha,\beta) \in \mathcal{E}$ with respect to P. □

5.3 When are the price processes martingales?

For each $\mathbb{G}$-predictable α and β satisfying conditions (23), let $Z(\alpha,\beta)$ be the P-martingale given by (22) and let $Q(\alpha,\beta)$ be the associated measure on $(\Omega,\mathcal{G}_T)$ equivalent to P. We fix again a horizon T such that $0 < T \le \bar T$ throughout.

The absence of arbitrage opportunities implies the existence of at least one martingale measure. In Theorem 5.1, we saw that each martingale measure $Q \in \mathcal{P}$ can be identified with some $Q(\alpha, \beta)$. We now explore necessary and sufficient conditions on (α, β) so that $Q(\alpha, \beta) \in \mathcal{P}$.

Theorem 5.3. *Consider the credit sensitive claim (T, c_T, R) where c_T is the payoff at time T if there is no default before T and R is the fraction of market value recovered in case of default. Then the claim's discounted price process is a martingale under $Q(\alpha, \beta)$ if and only if*

$$\mu_C(s) - \int_0^s (r + \sigma_C(u)\alpha_u)du = \int_0^{s\wedge\tau} (\beta_u + 1)(1 - R_u)dA_u^P \tag{34}$$

for $0 \le s \le T$.

Remark 5.4. *Equation (34) implies that the cumulative drift μ_C has continuous paths of finite variation that are not absolutely continuous with respect to the Lebesgue measure.*

Remark 5.5. *For the $\mathbb{G}$-predictable process β there exists an $\mathbb{F}$-predictable process $\bar{\beta}$ such that $\beta_t = \bar{\beta}_t$ on the set $\{t \le \tau\}$, cf. Jeulin & Yor (1978). Thus the integrand of the right hand side of equation (34) can be taken to be $\mathbb{F}$-adapted. In this situation, the $\mathcal{G}_s$-measurable random variable given by the left hand side of equation (34) takes the form $f(s \wedge \tau)$ where f is $\mathbb{F}$-adapted.*

Proof. From (21), it is equivalent to require that $(\bar{C}_t Z_t)_{t\le T}$, where $\bar{C}_t = C_t e^{-rt}$, is a $(P, \mathbb{G})$-martingale. From (28),

$$dZ_t = Z_{t-}(-\alpha_t dW_t + \beta_t dH_t). \tag{35}$$

Noting that $[C, e^{-r\cdot}] = 0$, we get by integration by parts

$$d\bar{C}_t = d(C_t \cdot e^{-rt}) = \bar{C}_{t-}(d\mu_C(t) - rdt + \sigma_C(t)dW_t - (1 - R_t)dN_t), \tag{36}$$

where we used the contingent claim dynamics described in A7.

Since the cumulative growth rate μ_C has paths of finite variation (A7), it is the difference between two increasing processes. Together with the fact that $\bar{C}_-$ is a positive process, this shows that the process defined by the Stieltjes integral $\int_0^t \bar{C}_{u-}d\mu_C(u)$ has paths of finite variation. Using this and Protter (2004, Chapter IV, Theorem 22), we have that

$$\begin{aligned}
[\bar{C}, Z]_t &= [\int_0^\cdot \bar{C}_{u-}(\sigma_C(u)dW_u - (1 - R_u)dN_u), \int_0^\cdot Z_{u-}(-\alpha_u dW_u + \beta_u dH_u)]_t \\
&= \int_0^t \bar{C}_{u-}Z_{u-}d[\sigma_C(u)W - (1 - R)N, -\alpha W + \beta H]_u \\
&= \int_0^t \bar{C}_{u-}Z_{u-}(-\sigma_C(u)\alpha_u du - \beta_u(1 - R_u)dN_u),
\end{aligned} \tag{37}$$

where we have used the fact that $[W, W]_t = t$, $[W, H] = 0$, and $[H, H] = [N, N] = [H, N] = N$. Integrating by parts, substituting equations (35), (36) and (37), and applying the Doob-Meyer decomposition $A^P_{\cdot\wedge\tau} + H = N$,

$$\begin{aligned} d(\bar{C}_t Z_t) =& \bar{C}_{t-} dZ_t + Z_{t-} d\bar{C}_t + d[\bar{C}, Z]_t \\ =& \bar{C}_{t-} Z_{t-} (d\mu_C(t) - (r + \sigma_C(t)\alpha_t)dt - (\beta_t + 1)(1 - R_t)dA^P_{t\wedge\tau} \\ &+ (\sigma_C(t) - \alpha_t)dW_t + (\beta_t R_t + R_t - 1)dH_t). \end{aligned} \tag{38}$$

We see that $(\bar{C}_t Z_t)_{t\le T}$ is a $(P, \mathbb{G})$-martingale if and only if the drift in (38) vanishes. This yields the condition

$$\mu_C(s) - \int_0^s (r + \sigma_C(u)\alpha_u)du - \int_0^{s\wedge\tau} (\beta_u + 1)(1 - R_u)dA^P_u = 0$$

for all $0 \le s \le T$, which completes the proof. □

6 Risk premia

As in the previous section, the reference measure is the physical measure P. We fix $T \in (0, \bar{T}]$. In Theorem 5.1, we show that each martingale measure $Q \in \mathcal{P}$ equivalent to P corresponds to a pair of $\mathbb{G}$-predictable processes α and β. In this section, we examine the relationship between these processes and the risk premia demanded by investors. The following result is a standard consequence of Girsanov's Theorem. We omit the proof.

Theorem 6.1. *Under the equivalent martingale measure $Q = Q(\alpha, \beta)$*

$$W^Q_t = W_t + \int_0^t \alpha_s ds, \quad t \le T,$$

is a $\mathbb{G}$-standard Brownian motion and

$$H^Q_t = N_t - \int_0^{t\wedge\tau} (1 + \beta_s)dA^P_s, \quad t \le T,$$

is a $\mathbb{G}$-martingale.

Remark 6.2. *Theorem 6.1 implies that the $\mathbb{G}$-compensator of the default process N under Q is given by*

$$A^Q_{t\wedge\tau} = \int_0^{t\wedge\tau} (1 + \beta_u)dA^P_u \tag{39}$$

where β is $\mathbb{F}$-predictable by Remark 5.5. This relationship extends to the trends under P and Q, which are both given by Definition 3.1. The random measures on $\mathbb{R}_+$ associated with the $\mathbb{F}$-predictable trends $A^P(\omega)$ and $A^Q(\omega)$ are both concentrated on the set

$\{t \geq 0 : V_t(\omega) = M_t(\omega)\}$. Hence, for almost all $\omega \in \Omega$, they are absolutely continuous with respect to each other. Theorem 68 in Dellacherie & Meyer (1982, Chapter VI) states that the corresponding density process γ is $\mathbb{F}$-predictable. Equation (39) implies that $\gamma_t = 1 + \beta_t$ on the set $\{t \leq \tau\}$. Using an argument similar to that used in the proof of Proposition 5.2, we can then show that $\gamma_t = 1 + \beta_t$ on $[0, \infty)$ and thus

$$A_t^Q = \int_0^t (1 + \beta_u) dA_u^P. \tag{40}$$

Remark 6.3. *In Theorem 5.1, we show that all equivalent martingale measures take the form $Q(\alpha, \beta)$. Consequently, every equivalent martingale measure can be viewed as an appropriate drift adjustment to the Brownian motion W and timing adjustment to the compensated jump martingale H.*

The interpretation of α and β as risk premia can be seen in the equity return dynamics under a martingale measure. We can write the equity return process in terms of both the physical P-dynamics (8) and the Q-dynamics

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= d\mu_S(t) + \sigma_S(t) dW_t - dN_t \\ &= d\mu_S^Q(t) + \sigma_S(t) dW_t^Q - dH_t^Q - dA_{t \wedge \tau}^Q. \end{aligned} \tag{41}$$

It follows from Theorem 6.1 that

$$\mu_S(t) - \mu_S^Q(t) = \int_0^t \alpha_u \sigma_S(u) du. \tag{42}$$

Equation (42) shows that the excess cumulative growth $p_S = \mu_S - \mu_S^Q$ on equity demanded by risk-averse equity investors has paths that are absolutely continuous with respect to Lebesgue measure. The excess growth rate $dp_S(t)/dt$ is proportional to the diffusive equity volatility $\sigma_S(t)$. The proportionality factor α_t equals the excess equity return per unit of diffusive equity risk. It can thus be interpreted as the market price of Brownian motion driven diffusion-type risk in firm values. The *diffusive risk premium* $\alpha \cdot \sigma_S$ is a $\mathbb{G}$-predictable stochastic process that depends on the underlying Brownian motion and the default time.

The default indicator is non-decreasing, hence a $(Q, \mathbb{G})$-submartingale. Theorem 6.1 defines its Doob-Meyer decomposition into a martingale H^Q and a non-decreasing predictable process $A_{\cdot \wedge \tau}^Q$ given by formula (39). The martingale property of H^Q implies the heuristic relation $dA_{t \wedge \tau}^Q = E^Q[dN_t \,|\, \mathcal{G}_t] = Q[dN_t = 1 \,|\, \mathcal{G}_t]$. With $r = 0$, the increment dA_t^Q can be interpreted as the pre-default price of an insurance contract that pays one dollar if default occurs over the next infinitesimal period of time, and zero otherwise. Equation (39) yields

$$\begin{aligned} dA_t^Q &= (1 + \beta_t) E^P[dN_t \,|\, \mathcal{G}_t] \\ &= (1 + \beta_t) P[t < \tau \leq t + dt \,|\, \mathcal{G}_t], \quad t < \tau. \end{aligned}$$

Hence, β_t provides the mapping between the instantaneous Q-default probability $Q[t < \tau \leq t + dt \,|\, \mathcal{G}_t]$ and the instantaneous P-default probability $P[t < \tau \leq t + dt \,|\, \mathcal{G}_t]$. This suggests the interpretation of β as the *default event risk premium* that is demanded by equity investors as compensation for assuming the risk of a downward jump due to default.

If investors are risk-neutral with respect to default event risk, they value default insurance with the physical P-default probability. Here the risk premium is zero, $\beta = 0$, and default probabilities are equal under P and Q. If investors love jump risk, their value of the insurance contract is lower than what is suggested by the P-default probability. The risk premium is negative, $\beta \in (-1, 0)$, and the Q-default probability is less than the physical default probability. If investors are averse to jump risk, they demand a spread for the default insurance contract that exceeds the physical default spread. The risk premium is positive, $\beta > 0$, and the Q-default probability exceeds the physical default probability.

Following El Karoui & Martellini (2001), we can define the associated market price of default event risk as $\log(1 + \beta)$. This is a $\mathbb{G}$-predictable process.

7 Valuing credit sensitive claims

According to assumption A6, a credit sensitive claim is characterized by a triple (T, c_T, R). Here, $c_T \in \mathcal{F}_T$ is the payoff at the horizon $T \in (0, \bar{T}]$ if there was no default, and $1 - R$ is the fractional loss in the pre-default market value of the claim if the firm defaults before T. For $Q \in \mathcal{P}$, a no-arbitrage, pre-default price $C_t = C_t(Q)$ of this claim at time $t \leq T$ is given by

$$C_t = E^Q[e^{-r(T-t)} c_T 1_{\{\tau > T\}} + e^{-r(\tau - t)} R_\tau C_{\tau-} 1_{\{\tau \leq T\}} \,|\, \mathcal{G}_t]. \tag{43}$$

This formula has the disadvantage of involving the default time τ explicitly. We prove an alternative reduced form representation of the claim price that is based on the Q-trend A^Q instead of τ, itself.

How can we obtain this trend? One approach is to reason as in the previous two sections. Assume that the reference measure π is the physical measure P. That is, assumptions A1-A8 hold under the actual P. We construct the P-trend A^P via Definition 3.1. Given the density $Z(\alpha, \beta)$, we then construct A^Q from A^P via (40). A second approach is to assume that the reference measure π is a martingale measure Q. This means that A1-A8 hold under Q. The trend A^Q can then be directly defined via Definition 3.1.

Consider the non-decreasing $\mathbb{F}$-predictable process $A^Q(R)$ defined by

$$A_t^Q(R) = \int_0^t (1 - R_s) dA_s^Q. \tag{44}$$

The following result suggests the term *positive-recovery trend* for $A^Q(R)$. Compare with Proposition 3.2.

Proposition 7.1. *For each $Q \in \mathcal{P}$, the process $H^Q(R)$ defined by*

$$H_t^Q(R) = (1 - R_\tau)N_t - A_{t\wedge\tau}^Q(R), \quad t \le T,$$

is a $(Q, \mathbb{G})$-martingale. In other words, $A_{\cdot\wedge\tau}^Q(R)$ is the $\mathbb{G}$-compensator of the $\mathbb{G}$-submartingale $(1 - R_\tau)N$ under Q.

Proof. The process $(1-R_\tau)N$ is zero before τ at which time it jumps to $(1-R_\tau) \in \mathcal{G}_\tau$ and stays there. Since R is $\mathbb{G}$-predictable, $(1 - R_\tau)N$ is $\mathbb{G}$-adapted. Since $A_{\cdot\wedge\tau}^Q(R)$ is clearly $\mathbb{G}$-adapted, so is the process $H^Q(R)$.

We show that $H^Q(R)Z$ is a P-martingale. We have

$$\begin{aligned}
[H^Q(R), Z]_t &= [(1 - R_\tau)N, \int_0^\cdot Z_{s-}(-\alpha_s dW_s + \beta_s dH_s)]_t \\
&= [\int_0^\cdot (1 - R_s)dN_s, \int_0^\cdot Z_{s-}\beta_s dN_s]_t \\
&= \int_0^t Z_{s-}\beta_s(1 - R_s)dN_s.
\end{aligned}$$

Integration by parts together with (39) and (44) yields that

$$\begin{aligned}
d(Z_t H_t^Q(R)) &= H_{t-}^Q(R)dZ_t + Z_{t-}((1 - R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) + Z_{t-}\beta_t(1 - R_t)dN_t \\
&= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)((1 - R_\tau)dN_t - (1 - R_t)dA_{t\wedge\tau}^P) \\
&= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)(1 - R_t)dH_t,
\end{aligned}$$

where H is the compensated jump P-martingale from the Doob-Meyer decomposition (14) of N. Since Z is also a P-martingale, $ZH^Q(R)$ is a P-martingale as well. This is equivalent to $H^Q(R)$ being a Q-martingale. □

We now turn to the representation of the claim price C in terms of $A^Q(R)$. Our result extends Duffie & Singleton (1999, Theorem 1) to the case where the continuous trend A^Q is not absolutely continuous, i.e. does not admit an intensity. This generalization is important in our setup as A^Q does *not* admit an intensity under assumptions A1-A8.

Theorem 7.2. *Suppose the trend A^Q is continuous. If the process Y given by*

$$Y_t = e^{-r(T-t)} E^Q[c_T e^{A_t^Q(R) - A_T^Q(R)} \,|\, \mathcal{G}_t], \quad t \le T, \tag{45}$$

has paths that are continuous at the default time, then the credit sensitive claim (T, c_T, R) admits an arbitrage-free value $C_t = Y_t(Q)$ on the no-default set $\{\tau > t\}$ at time $t \le T$.

Proof. First note that if A^Q is continuous, then $A^Q(R)$ is as well continuous. Let $K_t = e^{-rT}E^Q[c_T e^{-A_T^Q(R)} \,|\, \mathcal{G}_t]$ such that $Y_t = e^{A_t^Q(R)+rt}K_t$. Setting $\bar{Y}_t = Y_t e^{-rt} = e^{A_t^Q(R)}K_t$, by integration by parts we find that

$$d\bar{Y}_t = e^{A_t^Q(R)}dK_t + \bar{Y}_{t-}dA_t^Q(R).$$

We now define the process U by

$$U_t = (1 - N_t)\bar{Y}_t + \int_0^t R_s\bar{Y}_{s-}dN_s;$$

our goal is to show that $(U_t)_{t\leq T}$ is a Q-martingale. Since $\bar{Y}$ does not jump at τ by assumption, $\Delta\bar{Y}\Delta(1-N) = 0$ and we have by integration by parts

$$\begin{aligned}
dU_t &= d((1-N_t)\bar{Y}_t) + R_t\bar{Y}_{t-}dN_t \\
&= -\bar{Y}_{t-}dN_t + (1-N_{t-})d\bar{Y}_t + R_t\bar{Y}_{t-}dN_t \\
&= (1-N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}((1-R_t)dN_t - (1-N_{t-})dA_t^Q(R)) \\
&= (1-N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}((1-R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) \\
&= (1-N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}dH_t^Q(R),
\end{aligned}$$

for the Q-martingale $H^Q(R)$, cf. Proposition 7.1. Since K is also a Q-martingale, $(U_t)_{t\leq T}$ is as well a Q-martingale (all integrands are predictable and bounded for $T < \bar{T}$).

By the martingale property of U we get

$$(1-N_t)Y_t e^{-rt} = E^Q[(1-N_T)Y_T e^{-rT} + \int_t^T e^{-rs}R_sY_{s-}dN_s \,|\, \mathcal{G}_t].$$

Since $Y_T = c_T$, this yields

$$\begin{aligned}
(1-N_t)Y_t &= E^Q[(1-N_T)c_T e^{-r(T-t)} + \int_t^T e^{-r(s-t)}R_sY_{s-}dN_s \,|\, \mathcal{G}_t] \\
&= E^Q[1_{\{\tau>T\}}c_T e^{-r(T-t)} + e^{-r(\tau-t)}R_\tau Y_{\tau-}1_{\{t<\tau\leq T\}} \,|\, \mathcal{G}_t].
\end{aligned}$$

This implies

$$Y_t = E^Q[1_{\{\tau>T\}}c_T e^{-r(T-t)} + e^{-r(\tau-t)}R_\tau Y_{\tau-}1_{\{\tau\leq T\}} \,|\, \mathcal{G}_t]$$

on the set $\{\tau > t\}$, which is our assertion. □

The process C in Theorem 7.2 uniquely defines the price of the claim only if markets are complete. In the incomplete case, Theorem 7.2 leads to an interval of arbitrage-free prices for (T, c_T, R).

Bélanger et al. (2004) derive a similar price representation under a different set of assumptions. Becherer & Schweizer (2003) consider a situation in which default risk is driven by a fundamental traded asset, typically a stock, whose price jumps downward at default. Default times are intensity based and there is a mutual dependence between the intensity and the asset price process. Prices of contingent claims on the fundamental asset are represented as solutions to partial differential equations.

Under assumptions A1-A8, the pricing formula (45) can be simplified. The pre-default price (43), defined with respect to the investor filtration $\mathbb{G}$, is expressed in terms of an $\mathbb{F}$-conditional expectation. This is analogous to the formula (11) for the $\mathbb{G}$-conditional default probability.

Corollary 7.3. *Under assumptions A1-A8, the credit sensitive claim (T, c_T, R) admits an arbitrage-free value*

$$C_t = e^{-r(T-t)} E^Q[c_T e^{A_t^Q(R) - A_T^Q(R)} \,|\, \mathcal{F}_t] \tag{46}$$

on the no-default set $\{\tau > t\}$ at time $t \leq T$.

Proof. C3 implies that the continuous $\mathbb{F}$-martingale U defined by the $\mathcal{F}_t$-conditional expectation $U_t = E^Q[c_T e^{-A_T^Q(R)} \,|\, \mathcal{F}_t]$, $t \leq T$ is a $\mathbb{G}$-martingale. It follows that

$$U_t = E^Q[c_T e^{-A_T^Q(R)} \,|\, \mathcal{G}_t], \quad t \leq T,$$

so that the price process Y defined in Theorem 7.2 can be written as

$$Y_t = e^{-r(T-t)} E^Q[c_T e^{A_t^Q(R) - A_T^Q(R)} \,|\, \mathcal{F}_t].$$

Since the process defined by $e^{A_t^Q(R)}$ has continuous paths, Y has continuous paths as well. The result follows from Theorem 7.2. □

8 Toward calibrating credit

The output of a credit model is only as good as the data and methods used to calibrate it. The approach taken to calibration depends on the data that are available and the way in which the model is used. We conclude our article with a preliminary exploration of the issues surrounding calibration.

As discussed in the introduction, one main use of a credit model is to forecast the actual probability of default. If this is our only use for the model, a straightforward thing to do is to take the reference measure to be the physical measure P. This means that economic assumptions A1-A8 hold under P. As shown in Section 3, default probabilities can be calculated from the trend A^P via (11) and (17). The default probability is expressed in terms of the parameters μ and σ of the gross firm value process as well as parameters of the default barrier distribution G, which we assume is given by some suitable parametric model.

There is a standard approach to calibrating complete information structural default models with riskless interest rates, firm fundamental data and data from equity markets: see, for example, Duan (1994). We can extend this approach to the incomplete information case. However, many issues related to calibration and actual default probabilities are left open. One of the thorniest is related to the expected firm growth rate μ. Even for the simplest models such as Merton (1974), firm growth is notoriously difficult to estimate. In our framework, this estimation is complicated by the default event risk, which is neglected in many structural models. A resolution of these issues may lie in wealth of information embedded in bond and credit derivative markets.

This brings up the second purpose of a credit model, which is to price credit sensitive securities. If we are only interested in pricing securities, there is, a priori, no need to consider the physical measure P. Instead, we can directly model the trend A^Q under some pricing measure Q.

Suppose our calibration dataset consists of prices of credit sensitive claims issued by or referenced on a firm. Jarrow (2001) argues that both debt and equity prices should be taken into account here. We take the reference measure to be a pricing measure Q and assume A1-A8 hold under this Q. The trend A^Q can be calculated via (17). A parametric representation of the recovery process R^i for a traded credit sensitive claim i generates a parametric representation of the positive recovery trend $A^Q(R^i)$ using (44). This paves the way to calculating the model price C^i of that claim via Corollary 7.3. We can then fit the parameters $(\sigma, G, (R^i)_i)$, where G is again specified through some suitable parametric model. In this case, we minimize an objective function of the errors between model prices and market prices. This procedure identifies the chosen martingale measure Q as the equilibrium market pricing measure. This is important as we are not required to specify a particular martingale measure if markets are incomplete. Instead, we let the market choose Q.

To conclude, we take full advantage of the results in this article by calibrating simultaneously under the physical measure P and a martingale measure Q. As above, our procedure will identify Q with the equilibrium pricing measure. Further, a model simultaneously calibrated to P and Q has unrestricted use. It generates P-default probability forecasts and model prices of credit sensitive securities.

We outline the associated calibration procedure, which is described in detail in Giesecke & Goldberg (2004*a*). Once again, the reference measure is the physical measure P and the economic assumptions A1-A8 are associated to P. We calculate the trend A^P and P-default probabilities as discussed in Section 3. The associated positive recovery P-trend $A^P(R^i)$ is given by

$$A_t^P(R^i) = \int_0^t (1 - R_s^i) dA_s^P. \tag{47}$$

This quantity is analogous to the Q-quantity $A^Q(R^i)$ mentioned above. With the density $Z(\alpha, \beta)$ given by Theorem 5.1, the relationship (40) between the P- and Q-trends generalizes to positive recovery trends:

$$A_t^Q(R^i) = \int_0^t (1+\beta_s) dA_s^P(R^i). \tag{48}$$

When equations (47) and (48) hold, Corollary 7.3 yields the formula

$$\begin{aligned} C_t &= e^{-r(T-t)} E^Q[c_T e^{A_t^Q(R^i) - A_T^Q(R^i)} \,|\, \mathcal{F}_t] \\ &= e^{-r(T-t)} E^Q[c_T e^{-\int_t^T (1+\beta_s)(1-R_s^i) dA_s^P} \,|\, \mathcal{F}_t], \quad t \leq T. \end{aligned}$$

If β and R^i are constant, we get

$$C_t = e^{-r(T-t)} E^Q \left[c_T \left(\frac{G(M_T)}{G(M_t)} \right)^{(1+\beta)(1-R^i)} \Bigg| \, \mathcal{F}_t \right], \quad t \leq T.$$

We fit the parameters $(\mu, \sigma, G, (R^i)_i, \alpha, \beta)$ such that the error between model implied prices C^i and observed market prices, as well as the error between implied P-default probabilities and historical default probabilities is minimized. This problem is subject to the arbitrage conditions on α and β, see Section 5.3. This procedure estimates the equilibrium risk premia (α, β) and the associated equilibrium pricing measure $Q(\alpha, \beta)$.

A Probabilistic model structure

We detail the probabilistic structure underlying our model described in Section 2.

We introduce two probability spaces. The first is the filtered Wiener space $(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t\geq 0}, \pi_1)$ supporting the standard Brownian motion $\tilde{W}$. The second is the space $(\Omega_2, \mathcal{F}^2, \pi_2)$ supporting the random variable $\tilde{d}$. Here we may set $\Omega_2 = (0, X_0)$ for some constant $X_0 > 0$ and $\tilde{d} = \omega_2$ for $\omega_2 \in \Omega_2$, and $\mathcal{F}^2 = \sigma(\tilde{d})$. Our reference probability space is constructed as the product space

$$(\Omega, \mathcal{G}, \pi) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, \pi_1 \otimes \pi_2) \tag{49}$$

where the state of the world $\omega \in \Omega$ is the pair (ω_1, ω_2).

On this space, we define the standard Brownian motion $W(\omega) = \tilde{W}(\omega_1)$ and the random default barrier $d(\omega) = \tilde{d}(\omega_2)$ considered in assumptions A2 and A3, respectively. Notice that we do not observe ω_2, cf. assumption A5. Corresponding to assumption A3, we also introduce the random time τ by setting

$$\tau(\omega) = \inf\{t > 0 : V_t(\omega) \leq D(\omega)\}, \tag{50}$$

where $V_t = mt + \sigma W_t$ for constants $m \in \mathbb{R}$ and $\sigma > 0$ and $D = \log(d/X_0)$ is the normalized default barrier. The measure π_2 induces a distribution function G of the normalized barrier D via $\pi_2(0, X_0 e^x) = G(x)$ for all $x \in (-\infty, 0)$. The distribution function G is often called "prior." Letting $M_t(\omega) = \min_{s \leq t} V_s(\omega)$, we can write

$$\{\tau(\omega) > t\} = \{M_t(\omega) > D(\omega)\}. \quad (51)$$

Consider the standard filtration $\mathbb{F} = \mathcal{F}_{t\geq 0}$ generated by W on $(\Omega, \mathcal{G}, \pi)$. All the sets in $\mathbb{F}$ are of the form $F \times \Omega_2$ and $F \times \emptyset$ for $F \in \mathcal{F}_t^1$. Let $(\mathcal{S}_t)$ be the standard filtration generated by the indicator process $(1_{\{\tau \leq t\}})$. Corresponding to assumption A5, we can now introduce the enlarged filtration $\mathbb{G}$ on the reference space by setting $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{S}_t$.

The $(\pi, \mathbb{F})$-Brownian motion W is also a Brownian motion in the enlarged filtration $\mathbb{G}$. Indeed, because W ignores ω_2,

$$E^\pi[W_t | \mathcal{G}_s] = E^\pi[W_t | \mathcal{F}_s] = W_s, \quad t \geq s, \quad (52)$$

proving the martingale property of W in $\mathbb{G}$. Since the quadratic variation $[W, W]_t = t$ does not depend on the filtration, the result follows by Lévy's theorem.

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